Parallel Digital Currencies and Sticky Prices*

Harald Uhlig† Taojun Xie‡

December 23, 2020

Abstract

The recent rise of digital currencies opens the door to their use in parallel alongside official currencies ("dollar") for pricing and transactions. We construct a simple New Keynesian framework with parallel currencies as pricing units and sticky prices. Relative prices become a state variable. Exchange rate shocks can arise even without other sources of uncertainty. A one-time exchange rate appreciation for a parallel currency leads to persistent redistribution towards the dollar sector and dollar inflation. The share of the non-dollar sector increases when prices in the dollar sector become less sticky and when firms can choose the pricing currency.

Keywords: private money, cryptocurrency, digital currency, currency choice, monetary policy
JEL classification: E52, E30

*We thank participants at the CEPR-Bank of Finland Joint Conference 2020 for helpful comments.
†University of Chicago, NBER, CEPR, huhlig@uchicago.edu
‡Asia Competitiveness Institute, Lee Kuan Yew School of Public Policy, National University of Singapore, tjxie@nus.edu.sg

1 Introduction

The proliferation of cryptocurrencies and digital payment facilities has facilitated transactions in various currencies. Some currencies are pegged to the official one, such as bank deposits and stablecoins, while some are not, such as reward points and cryptocurrencies. Others such as Diem are somewhere in between. This raises several issues and concerns. In this paper, we examine the role of these currencies as a unit of account, i.e., the possibilities that firms may choose to price their products in one of several available currencies. With sticky prices in these currencies, new phenomena and implications for monetary policy arise. To examine these issues, we extend a basic New Keynesian (NK) framework as described in Galí (2015) by allowing for multiple parallel currencies. The classic exchange rate indeterminacy result by Kareken and Wallace (1981) applied to our framework then implies that exogenous exchange rate shocks can arise, even if there are no other sources of uncertainty. These exchange rate shocks in turn have real consequences due to the stickiness of prices.

Our focus lies in the macroeconomic implications and monetary policy consequences of these exchange rate shocks. On purpose, we take a bird's-eye view with respect to the underlying technology and abstract from otherwise important details such as blockchain implementation, consensus formation, reliability and regulation. While the analysis here may also be applied to the situation of dollarization in some developing countries, it is most suited to the examination of the future use of private cryptocurrencies in advanced countries: central banks there are committed to low inflation and the value of the private cryptocurrency may fluctuate for random reasons outside the control of some other government agency such as a foreign central bank.

By nature, this is a forward-looking paper. Currently, the regulatory framework for private cryptocurrencies is still under much debate, and regulators so far seem to push against their widespread introduction. The technology and inherent possibilities are intriguing enough, however, that more widespread adoption of private cryptocurrencies may well still come to pass. It is therefore important to examine the resulting macroeconomic effects and challenges to monetary policy ahead of time.

Our primary finding is that exchange rate shocks among these parallel currencies add a new source of macroeconomic volatility More specifically, a one-time appreciation of the non-official currency will lead to a persistent expansion in the sector using the official currency and a persistent decline in the nominal interest rate, with some caveats concerning the initial period and the details of the monetary policy rule. It is worth highlighting three key results. First, the nominal exchange rate between any pair of parallel currencies is a random-walk process, given that all currencies are perfect substitutes in providing liquidity services. Second, the price in each currency sector relative to the general price level is a state variable in an NK economy. This is a consequence of price rigidity that delays the process of prices converging to their desired levels after an exchange rate shock. The relative price also explains the persistent distributional output effects and the sectoral inflation dynamics. Third, an exchange rate shock is not neutral if it arises from a currency sector in which price adjustments are infrequent.

We finally endogenize the currency choice using a discrete choice model building on McFadden (1974) and examine the consequences of a change in price rigidity. We find that an increase of the price flexibility in the dollar sector leads to a persistent increase of the output in the non-dollar sector as a share of the economy, as well as vice versa. An flexibility increase in either sector induces a persistent fall in the nominal interest rate and aggregate inflation because it induces a persistent boost of the output gap.

Our paper advances the literature in a number of ways. Perhaps the paper closest to ours is Gopinath et al. (2020). Like us, they consider the issue of pricing in different currencies and the macroeconomic and monetary policy consequences in an NK framework. Different from them, we examine the consequences of parallel currencies within the same country rather than issues of international trade. The parallel currencies may not be under the control of a foreign central bank and may be subject to exogenous exchange rate shocks, in contrast to the analysis in Gopinath et al. (2020). To the best of our knowledge, we are the first to examine the consequences in the context of a New Keynesian model. Our analysis of the endogenous choice of a currency extends beyond Gopinath et al. (2010), allowing for random preferences and providing an interior solution rather than an either-or choice as in their paper.

More broadly, this paper contributes to are three strands of the literature. The most closely related is the emerging literature on the economics of private monies and cryptocurrencies. Following the rise of bitcoin and blockchain technology, the recent macroeconomic-oriented literature, such as Berentsen (1998), Chiu and Koeppl (2019), Garratt and Wallace (2018), Brunnermeier and Niepelt (2019), Fernández-Villaverde and Sanches (2019), Schilling and Uhlig (2019b), Schilling and Uhlig (2019a), Benigno (2019), Biais et al. (2020), Baughman and Flemming (2020) and Benigno et al. (2019), has analyzed cryptocurrencies by focusing on their role as a medium of exchange. By contrast, we focus on the unit-of-account role of currencies, i.e., their role in pricing decisions by producers. The literature on cryptocurrencies is considerably broader, of course, touching on many issues outside the analysis here: for an excellent book treatment, see Schär and Berentsen (2020). Asimakopoulos et al. (2019); Barrdear and Kumhof (2016); George et al. (2020); Grossa and Schillerb (2020); Ferrari et al. (2020) construct or estimate NK DSGE models with both private and central bank cryptocurrencies as competing media of exchange. Schilling and Uhlig (2019b) emphasize the random walk nature of exchange rates, building on the classic contributions by Kareken and Wallace (1981) and Manuelli and Peck (1990). This gives rise to the key shock that is the focus of much of our analysis. Our paper is related to the literature on currency substitution, with classic contributions by Girton and Roper (1981) and Matsuyama et al. (1993). Gopinath et al. (2010) examine endogenous choice between local currency pricing and producer currency pricing: we build on that analysis in Section 7. Like us, Adrian and Mancini-Griffoli (2019), Duffie (2019) and Brunnermeier et al. (2019) stress the increased currency competition arising from the digitalization of money. The third strand of the literature is the multi-sector NK framework such as Cienfuegos (2019), Barsky et al. (2007), Sterk (2010), Pasten et al. (2018) and Rubbo (2020). While these papers focus on sector-specific technology shocks and network effects in the presence of s a single currency, our focus is on exchange rate shocks in the presence of multiple currencies. For ease of exposition, we have chosen to abstract from network considerations in our analysis..

The remainder of the paper is organized as follows. Section 2 describes the model setup for our analyses. Section 3 presents the key linearized equations. Section 4 discusses two baseline cases, including dynamics under homogeneous price rigidity and with one flexible-price sector. Section 5 analyzes equilibrium dynamics under three alternative Taylor rules in detail. Section 6 relaxes the assumption of homogeneous price rigidity. Section 7 allows endogenous currency choice and examines transition dynamics as price rigidity changes. Section 8 concludes.

2 The Model

The model extends the basic NK framework presented in Chapter 3 of Galí (2015). We introduce two departures. First, multiple types of money provide liquidity services to households. This is modelled with a money-in-utility setup. Particularly, all monies are perfect substitutes in providing liquidity services. Second, different firms fix prices in different currencies. As a result, firms have to consider exchange rate dynamics when setting the optimal price, in contrast to a conventional NK setup. In consequence, the monetary policy rule can take a variety of forms.

2.1 Currencies and Price Indices

There is a total of J parallel currencies circulating in the economy. Each currency j has money supply $M_{j,t}$ in period t. Among them, there is one centralized currency, the money supply of which is managed by a central bank and $J-1$ currencies, which are created by private entities. Without loss of generality, the centralized currency is indexed by $j = 1$, and is named *dollar*. The price of currency j in terms of the dollar in period t or exchange rate is denoted by $\mathcal{E}_{j,t}$. The price of currency j in terms of a different currency j' can then be calculated as the ratio between $\mathcal{E}_{j,t}$ and $\mathcal{E}_{j',t}$. For formal reasons, we also define the exchange rate of a dollar to a dollar per $\mathcal{E}_{1,t} = 1$.

There is a continuum of goods $i \in [0, 1]$, each produced by a monopolistically competitive firm $i \in [0, 1]$. Each firm fixes its price in only one of the J currencies. Let $(V_{1,t}, \ldots, V_{J,t})$ be a partition of $[0, 1]$ and let $\upsilon_{j,t}$ be the measure of set $V_{j,t}$. We assume that firm $i \in V_{j,t}$ fixes or has fixed the price for its good in units of currency j, up to date t. For now, we shall consider this partition evolution as exogenously given, but return to the issue of currency choice in Section 7. The price set by firm i in currency j is denoted by $P_t(i)$. Despite the choice of pricing currency, the firm is always willing to accept any other parallel currency at the prevailing exchange rate. All firms $i \in V_{j,t}$ pricing in the same currency form a sector with the same index as the currency.

Define a sectoral price index for sector j, i.e., for all firms pricing in currency j, per

$$P_{j,t} \equiv \left[\frac{1}{\upsilon_{j,t}} \int_{V_{j,t}} P_t(i)^{1-\eta} di \right]^{\frac{1}{1-\eta}} \tag{1}$$

It follows that the general price index, expressed in terms of the dollar and defined per

$$P_t \equiv \left[\sum_{j=1}^{J} \int_{V_{j,t}} \left(\mathcal{E}_{j,t} P_t(i) \right)^{1-\eta} di \right]^{\frac{1}{1-\eta}} \tag{2}$$

is a composite of the sectoral price indices

$$P_t = \left[\sum_{j=1}^{J} \upsilon_{j,t} \left(\mathcal{E}_{j,t} P_{j,t} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}}. \tag{3}$$

We let

$$\Pi_t \equiv \frac{P_t}{P_{t-1}}$$

denote the general price inflation. Although the general price index and the dollar-sector price index are both expressed in terms of the dollar, their values are not necessarily the same. Only in the limiting case when all firms price in dollars, $\upsilon_1 \to 1$, the general price index is identical to the sectoral one, $P_t = P_{1,t}$. We define the relative price between the price in sector j and the general price level to be

$$\hat{P}_{j,t} = \frac{\mathcal{E}_{j,t} P_{j,t}}{P_t}, \tag{4}$$

which is an indicator of the respective currency's purchasing power, with a higher value corresponding to a weaker purchasing power. From the equation above, a weak purchasing power of currency j is associated with its depreciation and a higher sectoral price, or an appreciation and lower sectoral price of a different currency. The relative price $\hat{P}_{j,t}$ can also be interpreted as the real effective exchange rate of currency j. Similarly, we define the bilateral relative price $S_{jj',t}$ between two sectors j and j' as

$$S_{jj',t} = \frac{\mathcal{E}_{j,t} P_{j,t}}{\mathcal{E}_{j',t} P_{j',t}} \tag{5}$$

2.2 Households

A representative household's lifetime utility is a discounted flow of the period utility function of consumption bundle C_t, real money balances or liquidity L_t, and labor supply N_t, subject to an exogenous preference shock Z_t. We use a simple money-in-the-utility function for our purposes: the precise way win which money matters is not of particular importance for our analysis. Formally,

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, L_t, N_t\right) Z_t \tag{6}$$

where E_t is the expectation operator and $\beta < 1$ is the discount factor. The period utility function is

$$u\left(C_t, L_t, N_t\right) = \begin{cases} \frac{C_t^{1-\sigma}-1}{1-\sigma} + \frac{L_t^{1-\xi}-1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi} & \text{if} \quad \sigma \neq 1 \\ \log C_t + \frac{L_t^{1-\xi}-1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi} & \text{if} \quad \sigma = 1 \end{cases} \tag{7}$$

The consumption bundle C_t is an aggregation of the various consumption goods

$$C_t \equiv \left[\int C_t(i)^{1-\frac{1}{\eta}} di\right]^{\frac{\eta}{\eta-1}} = \left[\sum_{j=1}^{J} \int_{V_{j,t}} C_t(i)^{1-\frac{1}{\eta}} di\right]^{\frac{\eta}{\eta-1}} \tag{8}$$

where $C_t(i)$ is the consumption of variety i. The liquidity L_t is the real value of the nominal balances $M_{j,t}, j = 1, \ldots, J$, assumed to provide perfect substitutes in providing liquidity services:

$$L_t \equiv \sum_{j=1}^{J} L_{j,t} \quad \text{where} \quad L_{j,t} \equiv \frac{\mathcal{E}_{j,t} M_{j,t}}{P_t} \tag{9}$$

The representative household maximizes lifetime utility in Eq. (6) over choices $(C_t(i))_{i\in[0,1]}$, $(M_{j,t})_{j=1}^{J}$, B_t and N_t subject to the sequence of nominal period budget constraints

$$\sum_{j=1}^{J} \int_{V_{j,t}} \mathcal{E}_{j,t} P_t(i) C_t(i) di + B_t + \sum_{j=1}^{J} \mathcal{E}_{j,t} M_{j,t} = \exp(i_{t-1})\, B_{t-1} + \sum_{j=1}^{J} \mathcal{E}_{j,t} M_{j,t-1} + P_t W_t N_t + P_t \Gamma_t \tag{10}$$

where B_t is the holding of nominal government bonds at the end of period t, taking as given the dollar-denominated nominal rate of return i_t on bonds, the real wage W_t and the real dividends from the firms Γ_t.

The maximization problem of the household gives to the good-specific demand functions

$$C_t(i) = \left[\frac{\mathcal{E}_{j,t} P_t(i)}{P_t}\right]^{-\eta} C_t \quad \text{for} \quad i \in V_{j,t} \tag{11}$$

where we additionally needed to take the conversion into the official currency into account. With that, the budget constraint Eq. (10) can be written in terms of choices C_t, $(L_{j,t})_{j=1}^{J}$, B_t and N_t as

$$C_t + \frac{B_t}{P_t} + \sum_{j=1}^{J} L_{j,t} = \frac{\exp(i_{t-1})}{\Pi_t} \frac{B_{t-1}}{P_{t-1}} + \sum_{j=1}^{J} \frac{L_{j,t-1}}{\Pi_t} \frac{\mathcal{E}_{j,t}}{\mathcal{E}_{j,t-1}} + W_t N_t + \Gamma_t \tag{12}$$

2.3 Firms

Each firm i in the unit interval produces a differentiated good. The production function of a firm has the form

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{13}$$

where A_t is an exogenous level of technology common to all firms.

The price-setting process follows Calvo (1983). In each period t, each firm in set $V_{k,t-1}$ resets its price with probability $1-\theta_k$ and otherwise continues to be part of the set $V_{k,t}$. Given an opportunity to reset its price, assume that the firm i is now part of the set $V_{j,t}$, i.e., sets a new optimal price $P^*_{j,t}$ in currency j. This optimal price solves the profit maximization problem:

$$\max_{P^*_{j,t}} \sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[Q_{t,t+\ell} \left[\frac{\mathcal{E}_{j,t+\ell} P^*_{j,t}}{P_{t+k}} Y_{t+\ell}(i) - \Psi_{t+\ell} \left(Y_{t+\ell}(i) \right) \right] \right] \tag{14}$$

subject to the demand function

$$Y_{t+\ell}(i) = \left(\frac{\mathcal{E}_{j,t+\ell} P^*_{j,t}}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \tag{15}$$

where $Q_{t,t+\ell}$ is a stochastic discount factor, and $\Psi_{t+\ell}(\cdot)$ is the real total cost of production.

2.4 Monetary Policy

Monetary policy is governed by a Taylor rule in inflation and the output gap, where inflation and/or the output gap may refer to the economy as a whole or the dollar sector only. We state the precise formulation after loglinearizing our model, see subsections 3.4 and 3.5.

2.5 Equilibrium

In equilibrium, both the goods and labor markets clear. All goods produced by firm i are consumed by households:

$$Y_t(i) = C_t(i) \tag{16}$$

Define the aggregate output $Y_t = \left(\int_{[0,1]} Y_t(i)^{1-\frac{1}{\eta}} di \right)^{\frac{\eta}{\eta-1}}$. Then the aggregate output and consumption are equal:

$$Y_t = C_t. \tag{17}$$

Aggregating labor demand across all firms leads to

$$N_t^d = \int_0^1 N_t(i) di = \int_0^1 (Y_t(i)/A_t)^{\frac{1}{1-\alpha}} di = \left(\frac{Y_t D_t}{A_t} \right)^{\frac{1}{1-\alpha}} \tag{18}$$

where we define the price dispersion D_t as

$$D_t \equiv \left[\sum_{j=1}^{J} \int_{V_{j,t}} \left(\frac{\mathcal{E}_{j,t} P_t(i)}{P_t} \right)^{-\frac{\eta}{1-\alpha}} di \right]^{1-\alpha}$$

The labor market clearing condition is then

$$N_t = N_t^d \tag{19}$$

3 Linearization

We log-linearize the model presented in Section 2 around the zero-inflation steady state. Unless otherwise stated, lower cases are used to denote the deviations from the logarithmic steady states of the upper-cased variables. Detailed derivations are shown in Section B. For ease of exposition, we assume from now until Section 7 that the sets of firms $V_{j,t}$ pricing in currency j are independent of time, $V_{j,t} \equiv V_t$ and thus $\upsilon_{j,t} = \upsilon_j$. Section B contains more detailed derivations of the results here.

3.1 exchange rate dynamics

Proposition 1. *The exchange rate between any pair of parallel currencies j and j' follows a random-walk process:*

$$e_{j,t} - e_{j',t} = \mathrm{E}_t \left[e_{j,t+1} - e_{j',t+1} \right]. \tag{20}$$

Proof. See Section A.1. □

Proposition 1 is a version of the fundamental equation presented in Schilling and Uhlig (2019b) or the exchange rate indeterminacy equations in Kareken and Wallace (1981), Manuelli and Peck (1990) and Benigno et al. (2019), employing log-linearization and generalized to our context with any number of currencies. In particular, the exchange rate between currency j and the dollar is a martingale, up to first order:

$$e_{j,t} = \mathrm{E}_t \left[e_{j,t+1} \right]. \tag{21}$$

Without further restrictions, the martingale difference representing the exchange rate shock

$$\Delta e_{j,t} = e_{j,t} - e_{j,t-1}$$

is an exogenous random variable not usually present in New Keynesian macroeconomic models. The main focus of our analysis is to examine the macroeconomic and monetary policy consequences of these exchange rate shocks. Additional restrictions may arise from additional modelling assumptions. One possibility is that currency j is meant to represent a stablecoin with a consortium credibly maintaining a fixed exchange rate to the dollar, then $\Delta e_{j,t} \equiv 0$. Another possibility is that currency j is the official currency of a foreign central bank. We wish to keep the analysis general, however, and thus we shall treat $\Delta e_{j,t}$ as an exogenous random variable from here on.

3.2 Sectoral price dynamics and NKPC's

The following derivations are rather standard in the NK literature, but they need to be done with some care in order to keep track of where and how sectoral differences and exchange rates enter. It is thus worth stating the key details here and providing all remaining details in the appndix. In Section B, we show that, when prices are flexible in sector j, $\theta_j \to 0$, firms set prices to the desired levels given by

$$\tilde{p}_{j,t} = \Theta \, \mathrm{mc}_t + p_t - e_{j,t}, \tag{22}$$

where the log-linearized real marginal cost mc_t is given in Eq. (B.21) and where $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\eta}$. Use $\tilde{p}_t$ to represent the component independent of currency choice, $\tilde{p}_t \equiv \Theta \, \mathrm{mc}_t + p_t$. Eq. (22) shows that all desired prices are the same when converted into dollars:

$$\tilde{p}_{j,t} + e_{j,t} = \tilde{p}_t \quad \text{for all} \quad j = 1, ..., J \tag{23}$$

Eq. (22) together with Eq. (B.52) yields

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left[\left(\sigma + \frac{\alpha+\varphi}{1-\alpha} \right) y_t - \frac{\varphi+1}{1-\alpha} \, a_t \right] \tag{24}$$

where the relative price of sector j is defined as $\hat{p}_{j,t} \equiv p_{j,t} + e_{j,t} - p_t$. From Eq. (24), the price markup can be expressed in terms of the output gap and the relative price. When prices are flexible in all sectors, all firms price at the same desired level. In this case, $p_{j,t} = \tilde{p}_{j,t}$, $\hat{p}_{j,t} = 0$, and the aggregate output is at its natural level y_t^n. Eq. (24) becomes:

$$0 = -\Theta \left[\left(\sigma + \frac{\alpha+\varphi}{1-\alpha} \right) y_t^n - \frac{\varphi+1}{1-\alpha} \, a_t \right] \tag{25}$$

When prices are sticky, the first-order condition of the firm's profit-maximizing problem implies that the optimal price set by a firm pricing in currency j is given by a forward-looking function of future desired prices:

$$p^*_{j,t} = (1-\beta\theta_j)\sum_{\ell=0}^{\infty}(\beta\theta_j)^{\ell}\,\mathrm{E}_t\left[\tilde{p}_{j,t+\ell}\right] = (1-\beta\theta_j)\tilde{p}_{j,t} + \beta\theta_j E_t[p^*_{j,t+1}] \tag{26}$$

Using the fact that the sectoral price level is the weighted average between its past value and the optimal price, $p_{j,t} = \theta_j\, p_{j,t-1} + (1-\theta_j)\, p^*_{j,t}$, we can express the sectoral inflation in terms of its expected value one period ahead, and a deviation from the desired price:

$$\pi_{j,t} = \beta\,\mathrm{E}_t\left[\pi_{j,t+1}\right] - \lambda_j\,\left(p_{j,t} - \tilde{p}_{j,t}\right) \tag{27}$$

where

$$\lambda_j \equiv \frac{(1-\theta_j)\,(1-\beta\theta_j)}{\theta_j} \tag{28}$$

We note that λ_j is decreasing in θ_j.

Taking the difference between Eqs. (24) and (25) provides an expression for the price deviation in terms of the sector's relative price and the output gap:

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta\,\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)\,\tilde{y}_t \tag{29}$$

Plug this into Eq. (27). The NKPC for sector j is then an equation of its expected value for the next period, the output gap, and the sectoral relative price.

$$\pi_{j,t} = \beta\,\mathrm{E}_t\left[\pi_{j,t+1}\right] + \kappa_j\tilde{y}_t - \lambda_j\,\hat{p}_{j,t} \tag{30}$$

where

$$\kappa_j \equiv \lambda_j\,\Theta\,\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) \tag{31}$$

We note that κ_j is decreaasing in θ_j via λ_j. Two distinctive features are worth emphasizing. First, sectoral inflation is influenced by the aggregate output gap, not just the sectoral one. Second, there is an additional term on the relative price in the sectoral NKPC. Both features are signs of network effects across currency sectors.

3.3 Output-gap dynamics and the dynamic IS curve

The dynamic IS equation is the same as the one in a standard NK framework, which is derived from households' Euler equation and the market clearing condition:

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}\left[\hat{i}_t - \mathrm{E}_t[\pi_{t+1}] - \hat{r}_t^n\right], \tag{32}$$

where $\hat{i}_t$ is the deviation of the nominal interest rate from its steady state, and $\hat{r}_t^n$ is the natural rate of interest, which is a linear combination of productivity and preference shocks

$$\hat{r}_t^n = -\sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t. \tag{33}$$

The real interest rate is defined as the nominal interest rate adjusted for expected inflation:

$$\hat{r}_t = \hat{i}_t - \mathrm{E}_t[\pi_{t+1}]. \tag{34}$$

From the demand equations for sectoral consumption goods, the sectoral output is given by $y_{j,t} = -\eta\,\hat{p}_{j,t} + y_t$. With flexible prices in all sectors, $y_{j,t}^n = y_t^n$. When prices are rigid, the sectoral output gap is given by

$$\tilde{y}_{j,t} = -\eta\,\hat{p}_{j,t} + \tilde{y}_t \tag{35}$$

3.4 Key equations

We use bold fonts to symbolize $J \times 1$ vectors of sectoral parameters and variables. In particular,

$$\begin{aligned}
\boldsymbol{\upsilon} &\equiv [\upsilon_1, \upsilon_2, ..., \upsilon_J]' \\
\boldsymbol{\lambda} &\equiv [\lambda_1, \lambda_2, ..., \lambda_J]' \\
\boldsymbol{\kappa} &\equiv [\kappa_1, \kappa_2, ..., \kappa_J]' \\
\boldsymbol{\pi}_t &\equiv [\pi_{1,t}, \pi_{2,t}, ..., \pi_{J,t}]' \\
\hat{\boldsymbol{p}}_t &\equiv [\hat{p}_{1,t}, \hat{p}_{2,t}, ..., \hat{p}_{J,t}]' \\
\tilde{\boldsymbol{y}}_t &\equiv [\tilde{y}_{1,t}, \tilde{y}_{2,t}, ..., \tilde{y}_{J,t}]' \\
\boldsymbol{\Delta e}_t &\equiv [\Delta e_{1,t}, \Delta e_{2,t}, ..., \Delta e_{J,t}]'
\end{aligned}$$

From its definition, the general price inflation can be expressed as a vector product between the sizes of the currency sectors and the exchange rate-adjusted sectoral inflation:

$$\pi_t = \boldsymbol{\upsilon}' \left(\boldsymbol{\pi}_t + \boldsymbol{\Delta e}_t \right). \tag{36}$$

The random-walk process of the exchange rates Eq. (21) implies that the vector of expected value in the next period is $\mathrm{E}_t \left[\Delta \boldsymbol{e}_{t+1} \right] = \mathbf{0}$, so that the one-period-ahead expected inflation is

$$\mathrm{E}_t \left[\pi_{t+1} \right] = \boldsymbol{\upsilon}' \mathrm{E}_t \left[\boldsymbol{\pi}_{t+1} \right]. \tag{37}$$

We can then express our NK framework with J parallel currencies using the following $(3J+2)$-equation system, which includes a general Taylor rule

$$\tilde{y}_t = \mathrm{E}_t \left[\tilde{y}_{t+1} \right] - \sigma^{-1} \left(\hat{i}_t - \boldsymbol{\upsilon}' \, \mathrm{E}_t \left[\boldsymbol{\pi}_{t+1} \right] - r_t^n \right) \tag{38}$$

$$\boldsymbol{\pi}_t = \beta \, \mathrm{E}_t \left[\boldsymbol{\pi}_{t+1} \right] + \boldsymbol{\kappa} \, \tilde{y}_t - \boldsymbol{\lambda} \circ \hat{\boldsymbol{p}}_t \tag{39}$$

$$\tilde{\boldsymbol{y}}_t = -\eta \hat{\boldsymbol{p}}_t + \tilde{y}_t \tag{40}$$

$$\hat{\boldsymbol{p}}_t = \hat{\boldsymbol{p}}_{t-1} + \left(\mathbf{I} - \mathbf{1} \, \boldsymbol{\upsilon}' \right) \left(\boldsymbol{\pi}_t + \Delta \boldsymbol{e}_t \right) \tag{41}$$

$$\hat{i}_t = \boldsymbol{\Phi}'_{\boldsymbol{\pi}} \left(\boldsymbol{\pi}_t + \boldsymbol{\Delta e}_t \right) + \boldsymbol{\Phi}'_{\boldsymbol{y}} \, \tilde{\boldsymbol{y}}_t \tag{42}$$

where $\circ$ is an operator for element-wise multiplication and where $\boldsymbol{\Phi}_{\boldsymbol{\pi}}$ and $\boldsymbol{\Phi}_{\boldsymbol{y}}$ are the vectors of Taylor rule cofficients on the various sectoral inflation rates and output gaps. We note that Eq. (41) implies that $\boldsymbol{\upsilon}' \hat{\boldsymbol{p}}_t = 0$, provided this was true in $t-1$. We shall assume so throughout. With that, Eq. (40) implies $\boldsymbol{\upsilon}' \tilde{\boldsymbol{y}}_t = \tilde{y}_t$.

The main difference of this NK framework with J parallel currencies from the one in Galí (2015) lies in the fact in that an exchange rate shock in any sector spills over to the aggregate economy. An unexpected one-time appreciation of currency j leads to a higher price in sector j and a higher general price level. The relative price is higher in sector j, but those in all the other sectors are lower, as seen from Eq. (41). Demand for goods sold by sector j is now relatively lower, leading to lower inflation in sector j but higher inflation elsewhere.

In addition, Eq. (41) shows that a one-time exchange rate shock[1] has nonetheless persistent effects. Due to the infrequent price adjustments, in each period, only a fraction of the firms can optimize their prices in response to the exchange rate shock. Therefore deviations in relative prices decline only gradually over time, as firms reset their prices.

[1] A one-time exchange rate shock leads to a permanent change in the exchange rate.

These key equations are similar to the NK framework with a production network presented in Cienfuegos (2019). In particular, the relative prices are state variables of the economy. Although we do not discuss the production network in this paper, an exchange rate shock also leads to a network effect on the relative prices of all sectors of the economy.

From Eqs. (36) and (39), we derive the generalized aggregate inflation:

$$\pi_t = \beta\, \mathrm{E}_t\left[\pi_{t+1}\right] + \boldsymbol{\upsilon}'\boldsymbol{\kappa}\,\tilde{y}_t - \boldsymbol{\upsilon}'\left(\boldsymbol{\lambda}\circ\hat{\boldsymbol{p}}_t\right) + \boldsymbol{\upsilon}'\Delta\boldsymbol{e}_t \tag{43}$$

Both the sectoral relative prices and the exchange rates influence aggregate inflation. The extents of such influences are contingent on the sizes of the respective currency sectors.

3.5 Three Taylor Rules: AIAO, DIAO and DIDO.

Equation (42) allows for considerable flexibility in specifying the monetary policy rule. We shall focus on three variants in particular. The first variant entails a nominal interest rate responding to Aggregate Inflation and the Aggregate Output gap (AIAO) per $\boldsymbol{\Phi}'_{\boldsymbol{\pi}} = \phi_\pi \boldsymbol{\upsilon}'$ and $\boldsymbol{\Phi}'_{\boldsymbol{y}} = \phi_y \boldsymbol{\upsilon}'$ for some $\phi_\pi \in \mathbf{R}, \phi_y \in \mathbf{R}$. We shall show that this NK framework differs from the one in the standard literature only in the existence of an exchange rate disturbance in the NKPC, under the assumption of homogeneous price rigidity; see Section 5. For that reason, we choose this to be our baseline Taylor rule and benchmark.

In the second variant, the nominal interest rate responds to the Dollar-sector Inflation and Aggregate Output gap (DIAO), where now $\boldsymbol{\Phi}'_{\boldsymbol{\pi}} = \phi_\pi[1, 0, 0, ..., 0]$. This may be a more appealing choice, as it makes the central bank responsible for the dollar inflation, as often enshrined by law, but concerned with the whole economy. The last variant features a nominal interest rate responding to the Dollar-sector Inflation and Dollar-sector Output (DIDO), where now also $\boldsymbol{\Phi}'_{\boldsymbol{y}} = \phi_y[1, 0, 0, ..., 0]$. This assumes that the central bank only responds to economic dynamics in the dollar sector. This can be justified if the non-Dollar sector is considered to be a shadow economy outside a legal or regulatory framework, or if law makers and the central bank simply decide to not concern themselves with macroeconomic developments outside the dollar-denominated sector.

4 Baseline Cases

We present two baseline cases with regard to the impacts of an exchange rate shock. The first one is when price rigidity is homogeneous across all currency sectors. In such a case, the model economy deviates from a conventional one only in the presence of parallel currencies, hence the existence of exchange rates. We are then able to focus exclusively on the role of the exchange rate shock in driving economic dynamics. The second case is when prices are flexible in one currency sector while being rigid in the others. This case presents a special scenario in which the exchange rate shock's impacts are muted. In other words, the exchange rate shock is neutral.

4.1 Homogeneous price rigidity

We summarize the bilateral associations in an economy with homogeneous price rigidity with the following proposition. Let $s_{jj',t}$ denote the log-deviation of the relative price $S_{jj',t}$ defined in Eq. (5).

Proposition 2. *Between any two sectors j and j' with homogeneous price rigidity θ,*

1. *the optimal prices in both sectors are equivalent, $p^*_{j,t} + e_{j,t} = p^*_{j',t} + e_{j',t}$;*
2. *the bilateral relative price is an autoregressive process, $s_{jj',t} = \theta\,(s_{jj',t-1} + \Delta e_{j,t} - \Delta e_{j',t})$;*
3. *the inflation differential is linear in the bilateral relative price, $\pi_{j,t} - \pi_{j',t} = -\frac{1-\theta}{\theta}\, s_{jj',t}$;*
4. *the output-gap differential is linear in the bilateral relative price, $\tilde{y}_{j,t} - \tilde{y}_{j',t} = -\eta\, s_{jj',t}$.*

Proof. See Section A.2. □

The first result is parallel to Proposition 1 of Gopinath et al. (2010), which states that local currency pricing and producer currency pricing are equivalent. We argue here that while different choices of pricing currencies end up with an equivalent price in the baseline case of homogeneous price rigidity, this result may not hold when price rigidities differ across the currency sectors, except when the expected desired price level is constant.

The second to the fourth results state that the bilateral relative price is a state variable for inter-sector differentials in inflation and output gaps. These differentials arise regardless of the choice of monetary policy, as no assumption on the monetary policy is needed to arrive at these conclusions. Instead, price rigidity is the only characteristic of the economy that causes the different inflation dynamics between any pair of currency sectors. The output gap differential is influenced by the elasticity of substitution, in addition to price rigidity. From the negative signs, the sector that experiences a currency appreciation always produces less output and has lower inflation, as compared to a sector with no currency appreciation.

Two scenarios are relevant to Proposition 2. The first scenario is when currency j experiences an appreciation, while currency j' does not. In the period of an unexpected appreciation of currency j, the dollar-denominated prices of sector-j products deviate above their desired levels. The price in sector j, relative to that in sector j', becomes higher. Demand for the sectoral goods changes as it is sensitive to relative prices. The second scenario is when both currencies j and j' do not experience an appreciation. Suppose that the exchange rate shock does not arise from either of the two currencies, but a third currency. The inflation and output-gap dynamics are identical between sectors j and j'.

Extending the assumption of homogeneous price rigidity to the aggregate economy, we arrive at an NKPC that is similar to the one in Galí (2015). We present this NKPC in the following proposition.

Proposition 3. *The new Keynesian Phillips curve for aggregate inflation is independent of the relative price dynamics if price rigidity is homogeneous across all currency sectors:*

$$\pi_t = \beta \, \mathrm{E}_t \left[\pi_{t+1}\right] + \kappa \, \tilde{y}_t + \boldsymbol{v}' \Delta \boldsymbol{e}_t \tag{44}$$

Proof. See Section A.3. □

Proposition 3 posits that when price rigidity is homogeneous, the net effect of sectoral prices on aggregate inflation is zero.

4.2 Dynamics in a flexible sector

The second baseline case we analyse considers a currency sector with flexible prices among other currency sectors with the same degree of price rigidity. In this economy, it matters whether an exchange rate shock arises from the currency sector with flexible prices or a sector with infrequent price adjustments.

Proposition 4. *An exchange rate shock to a non-official currency j does not spill over to the other currency sectors if prices are flexible in sector j.*

Proof. See Section A.4. □

When the shock originates from the sector with flexible prices, the price tends to deviate from the steady state, which is the desired price. Firms in this sector adjust their prices so that the effect of the exchange rate shock is offset. As a result, the dollar-denominated price remains the same as before the exchange rate shock. Hence, the relative price is unchanged. There is no change in macroeconomic dynamics. The exchange rate shock is therefore neutral.

On the other hand, an exchange rate shock leads to economy-wide responses as long as it arises from a sector with sticky prices. The flexible sector does not remain unchanged when there is an exchange rate shock from another sector with price rigidity. Due to the infrequent adjustment of prices, there are changes in the aggregate price level and the output gap as firms adjust their prices toward the desired level. As a result, the desired price varies. Since firms in this sector always price at the desired level, inflation and the output gap vary, following the changes in the desired price.

5 Equilibrium Dynamics: Homogeneous Price Rigidity

With the linear NK framework laid out in the previous section, we can analyze the economic dynamics. Our analyses involve the responses of output gaps and inflation to an unexpected exchange rate shock.

To simplify the exposition, we focus on a two-sector economy $J = 2$. As in the previous section, firms in sector 1 price in dollars, and firms in sector 2 price in a non-dollar currency. This allows us to simplify the notation as follows. Let the size of the non-dollar sector be $\upsilon_2 = \upsilon$. Then the size of the dollar sector is $\upsilon_1 = 1-\upsilon$. The exchange rate of the dollar is normalised to $e_{1,t} = 0$, and we let the exchange rate of the alternative currency be $e_{2,t} = e_t$. It is also convenient to drop the currency indices in the bilateral relative price of the two sectors so that $s_t \equiv \hat{p}_{2,t} - \hat{p}_{1,t}$. From Proposition 2, the law of motion of the bilateral relative price follows an autoregressive process

$$s_t = \theta \left(s_{t-1} + \Delta e_t\right) \tag{45}$$

where $\Delta e_t \sim N(0, \sigma^2_{\Delta e})$ is the residual of the random-walk process in Proposition 1. It follows from the definition of the general price index, $(1-\upsilon)\,\hat{p}_{1,t} + \upsilon\,\hat{p}_{2,t} = 0$, that the relative prices in the respective sectors can be expressed in terms of s_t, so $\hat{p}_{1,t} = -\upsilon s_t$ and $\hat{p}_{2,t} = (1-\upsilon)\,s_t$. With homogeneous price rigidity across the two sectors, $\theta_1 = \theta_2 = \theta$, the subscripts of the parameters κ_1, κ_2, λ_1, and λ_2 can be dropped. The sectoral NKPCs are expressed in terms of the output gap and the bilateral relative price as:

$$\pi_{1,t} = \beta\,\mathrm{E}_t[\pi_{1,t+1}] + \kappa\,\tilde{y}_t + \lambda\,\upsilon\,s_t \tag{46}$$

$$\pi_{2,t} = \beta\,\mathrm{E}_t[\pi_{2,t+1}] + \kappa\,\tilde{y}_t - \lambda\,(1-\upsilon)\,s_t \tag{47}$$

The model parameters are summarized in Table 1. Most of the parameters follow Galí (2015). The size of the non-dollar sector, υ, and the standard deviation of the exchange

Table 1: Parameter values in benchmark model.

Parameter	Value	Description
α	0.250	Share of labor input in production function
σ	1.000	Coefficient of risk aversion
φ	5.000	Inverse Frisch elasticity of labor supply
β	0.990	Discount factor
θ_1	0.750	Probability of not adjusting prices in dollar sector
θ_2	0.750	Probability of not adjusting prices in non-dollar sector
η	9.000	Elasticity of substitution among consumption goods
ϕ_π	1.500	Interest-rate reaction to inflation
ϕ_y	0.125	Interest-rate reaction to output gap
υ	0.200	Size of non-dollar sector
$\sigma_{\Delta e}$	0.250	Standard deviation of exchange-rate shock

Note: All parameters, except υ and $\sigma_{\Delta e}$, are obtained from Galí (2015).

rate shock, $\sigma_{\Delta e}$, are new. We let the non-dollar sector take 20% of the market. We pick the size of a one-standard-deviation exchange rate shock to be comparable to a monetary policy shock in Galí (2015) and set it to 0.25%.

5.1 Aggregate inflation, aggregate output gap

Using Proposition 3, the linearized NK framework describing the dynamics of the aggregate output gap and inflation, including the baseline AIAO Taylor rule, condenses to a three-equation system as follows

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1}\left(\hat{i}_t - \mathrm{E}_t[\pi_{t+1}] - \hat{r}_t^n\right) \tag{48}$$

$$\pi_t = \beta\,\mathrm{E}_t[\pi_{t+1}] + \kappa\tilde{y}_t + \upsilon\,\Delta e_t \tag{49}$$

$$\hat{i}_t = \phi_\pi\,\pi_t + \phi_y\,\tilde{y}_t \tag{AIAO}$$

The above equations differ from a standard NK framework only in the exchange rate shock in the NKPC. The additional term in the NKPC here implies that the path of aggregate inflation is influenced by both the size of the non-dollar sector and the standard deviation of the exchange rate shock. Since the nominal interest rate responds to aggregate inflation and the aggregate output gap, the exchange rate shock, therefore, has a direct impact on the nominal interest rate.

As all exogenous shocks are assumed to be uncorrelated, the economic dynamics under a productivity shock and a preference shock are identical to those in Galí (2015). We therefore focus only on the exchange rate shock. Without productivity and preference shocks, the natural rate of interest in the dynamic IS curve vanishes . In Fig. 1, the

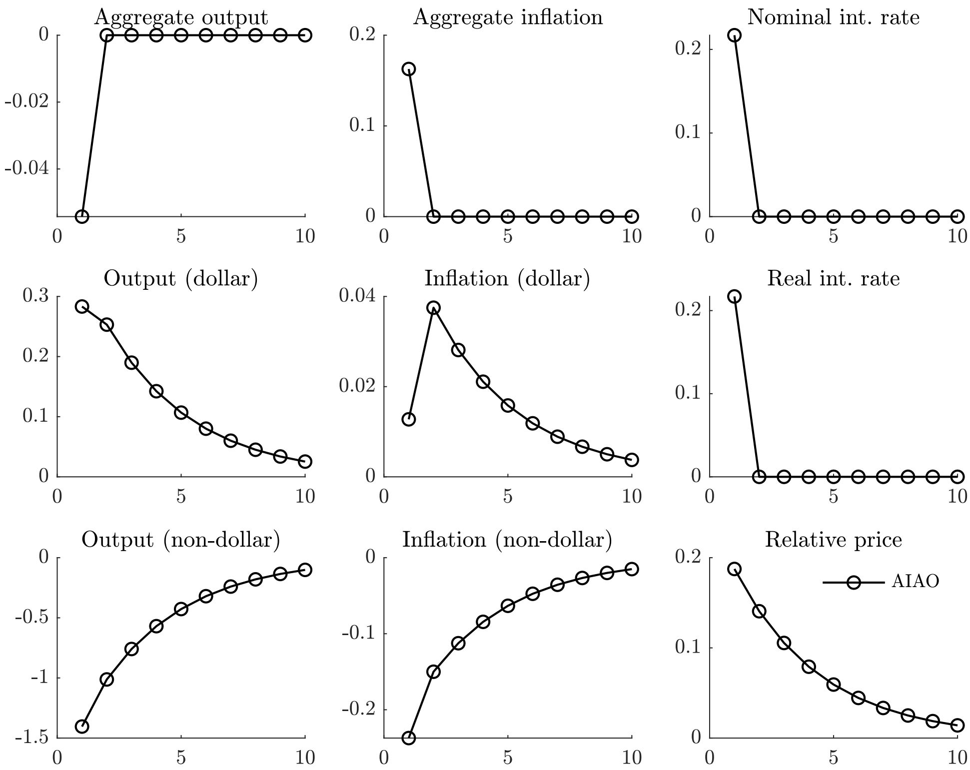

Figure 1: Impulse responses to a 0.25% appreciation of the exchange rate. Under the AIAO regime, the nominal interest rate responds to the aggregate inflation and the aggregate output gap. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange rate shock.

impulse responses to a *25-basis-point nominal appreciation in the non-dollar currency* are presented. We discuss the impulse responses in detail next.

5.1.1 Aggregate dynamics

The three plots in the first row of Fig. 1 depict that the aggregate output gap, aggregate inflation, and the nominal interest rate return to their steady states immediately after the period of the exchange rate shock. To see the analytical solution, substitute the Taylor rule Eq. (AIAO) into the dynamic IS curve, and use the method of undetermined coefficients to solve for the paths of aggregate inflation and the aggregate output gap. The impulse responses as functions to the exchange rate shock are

$$\hat{i}_t = \upsilon\,\sigma\,\phi_\pi\,\Omega\,\Delta e_t. \tag{50}$$

$$\tilde{y}_t = -\upsilon\,\phi_\pi\,\Omega\,\Delta e_t \tag{51}$$

$$\pi_t = \upsilon\,\left(\sigma + \phi_y\right)\Omega\,\Delta e_t, \tag{52}$$

where

$$\Omega \equiv \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} > 0 \tag{53}$$

We note that Ω is increasing in θ via κ. As the equations show, an unexpected one-time appreciation of the non-dollar currency unambiguously induces a one-time contemporaneous negative response in the output gap, and a one-time contemporaneous positive response in aggregate inflation. This is because the exchange rate shock leads to a higher consumer price and hence lower aggregate demand. The resulting effect on monetary policy is a contractionary one, as seen from the positive coefficient in equation (50) for the nominal interest rate. The random-walk process of the exchange rate means that $\mathrm{E}_t[\pi_{t+1}] = 0$, and hence the real interest rate coincides with the nominal interest rate.

Note that the responses of the aggregate variables are proportional to the size of the non-linear sector. When few firms opt to price in the non-dollar currency, $\upsilon \to 0$, an unexpected exchange rate shock has negligible influence on the aggregate economic dynamics. Besides, the fact that the coefficient Ω increases in θ means that higher price rigidity implies higher responsiveness of the aggregate variables to an exchange rate shock.

5.1.2 Sectoral dynamics

We now examine the sectoral dynamics so as to analyze the interactions between the dollar and the non-dollar sectors. Proposition 2 has shown that the bilateral differentials in sectoral prices, inflation, and output gaps are results of price rigidity and imperfect substitution among the goods. The inflation and output-gap differentials are both linear in the bilateral relative price:

$$\pi_{2,t} - \pi_{1,t} = -\frac{1-\theta}{\theta} s_t. \tag{54}$$

$$\tilde{y}_{2,t} - \tilde{y}_{1,t} = -\eta\, s_t. \tag{55}$$

The negative signs in Eqs. (54) and (55) imply that both the output gap and inflation are higher in the dollar sector compared to those in the non-dollar sector, given an appreciation of the non-dollar currency. The effect is persistent due to the persistence of s_t, see equation (45). Higher price stickiness θ implies more persistence as well as a smaller difference between the sectoral inflation rates.

Substituting the aggregate output gap into the sectoral output gap functions in

Eq. (35) gives the following dynamics of the sectoral output gaps:

$$\tilde{y}_{1,t} = \upsilon\, \eta\, s_t - \upsilon\, \phi_\pi\, \Omega\, \Delta e_t \tag{56}$$

$$\tilde{y}_{2,t} = -\,(1-\upsilon)\; \eta\, s_t - \upsilon\, \phi_\pi\, \Omega\, \Delta e_t \tag{57}$$

These sectoral movements are shown in the second and third rows of Fig. 1. From the impulse responses, there are distributional effects between the two sectors. Output and inflation in the dollar sector are positive, while output and inflation in the non-dollar sector are negative, for the chosen parameters. While this is true generally in all periods after the initial exchange rate shock, this is not necessarily so in that initial period, as a closer inspection of the two equations above will now show.

The first components of the sectoral output gaps are the substitution effects arising from a change in the relative sectoral price. The second components are the income effects due to lower aggregate output as seen in Eq. (51). The contemporaneous responses to an unexpected appreciation in the non-dollar currency are net outcomes of both the income effect and the substitution effect. The income effect drives the sectoral outputs below their natural level. The substitution effect, on the other hand, causes households to consume relatively more goods from the dollar sector and less from the non-dollar sector. Both effects result in lower output in the non-dollar sector, but in the dollar sector, the resulting output depends on which effect is dominant. Substitute Eq. (45) into Eq. (56) and combine the coefficients of the exchange rate shock. The dollar-sector firms produce above the natural level of output during the period of the exchange rate shock if:

$$\frac{1}{\eta\,\theta} - \kappa < \frac{\sigma + \phi_y}{\phi_\pi} \tag{58}$$

in which case the substitution effect dominates the income effect. The right-hand side is an indicator of the nominal interest rate's responsiveness to aggregate inflation and the output gap. A greater value on the right-hand side is due to a greater response to the aggregate output gap and/or a smaller response to aggregate inflation.

The inequality also implies that an unexpected exchange rate shock may cause the dollar-sector firms to produce below the natural level in the initial period of the exchange rate appreciation, when either the elasticity of substitution among consumer goods is sufficiently low (η is sufficiently small), or the interest rate is sufficiently responsive to aggregate inflation (ϕ_π is sufficiently large), and the interest rate is not responsive enough to the output gap (ϕ_y is sufficiently small). In the first instance, the small substitution elasticity limits households' willingness to consume more dollar-sector goods and less non-dollar-sector goods. In the second instance, the decline in the nominal interest rate is small, limiting the income effect of the exchange rate shock. The degree of price rigidity, however, has no clear influence on the direction of response.

In the periods following the exchange rate shock, dollar-sector output is persistently above the natural level, while non-dollar-sector output is persistently below it. This is because the exchange rate shock no longer causes a change in aggregate inflation. The income effect is not present after the period of the shock. The national income is back at its natural level. Due to price rigidity, the substitution effect, however, remains until the bilateral relative price returns to its steady state. Because the income effect is only one-off, kinks are expected in the impulse responses in the second period.

Using the method of undetermined coefficients, one can solve for sectoral inflation in terms of the exchange rate shock and the relative price as:

$$\pi_{1,t} = \frac{\upsilon\,(1-\theta)}{\theta} s_t - \upsilon\,\kappa\,\phi_\pi\,\Omega\,\Delta e_t \tag{59}$$

$$\pi_{2,t} = -\frac{(1-\upsilon)\,(1-\theta)}{\theta} s_t - \upsilon\,\kappa\,\phi_\pi\,\Omega\,\Delta e_t \tag{60}$$

As in the case of output responses, the exchange rate shock causes aggregate demand to be lower, and hence puts downward pressure on inflation. The substitution effect results in a demand-pulled inflation in the dollar sector. Firms adjust their price when opportunities arise. The rate at which inflation changes is subject to the degree of price rigidity. Substitute Eq. (45) into Eq. (59) and combine the coefficients of the exchange rate shock. Inflation in the dollar sector during the period of the exchange rate shock is higher if

$$(1-\beta\,\theta)\,\Theta\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) < \frac{\sigma+\phi_y}{\phi_\pi}. \tag{61}$$

This inequality also implies that, in the period of an exchange rate shock, the dollar sector may respond with lower inflation when either the degree of price rigidity is sufficiently low (θ is sufficiently small), or the interest rate is sufficiently responsive to aggregate inflation (ϕ_π is sufficiently large), and less responsive to the aggregate output gap (ϕ_y is sufficiently small). In the non-dollar sector, inflation is always below the steady state, as firms need to offset the price hike due to the exchange rate shock. In the periods after the exchange rate shock, the inflation dynamics are linear in the bilateral relative price.

It is of policy interest to find from Eqs. (56) and (59) that the dollar-sector variables are proportional to the size of the non-dollar sector. This is similar to the aggregate variables. In the case where no firm prices in the non-dollar currency, the dollar-sector variables do not respond to an exchange rate shock.

5.2 Alternative Taylor rules

For the other two variants of the Taylor rule, the nominal interest rate responds to the dollar-sector inflation instead of the aggregate one. Under such monetary policy rules, the exchange rate shock does not influence the nominal interest rate directly, but via the bilateral relative price, which causes changes in the dollar-sector inflation. The NK framework can be rearranged to a three-equation system with the aggregate output gap, the dollar-sector inflation, the nominal interest rate, and the bilateral relative price as the endogenous variables. The non-policy block consists of the following dynamic IS curve and dollar-sector NKPC, in addition to Eq. (45):

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1} \left[\hat{i}_t - \mathrm{E}_t[\pi_{1,t+1}] + \upsilon\ (1-\theta)\ s_t - \hat{r}_t^n\right] \tag{62}$$

$$\pi_{1,t} = \beta\, \mathrm{E}_t[\pi_{1,t+1}] + \kappa\, \tilde{y}_t + \lambda\, \upsilon\, s_t. \tag{63}$$

We consider the following two variants of the Taylor rule:

$$\hat{i}_t = \phi_\pi\, \pi_{1,t} + \phi_y\, \tilde{y}_t \tag{DIAO}$$

$$\hat{i}_t = \phi_\pi\, \pi_{1,t} + \phi_y\ (\tilde{y}_t + \eta\, \upsilon\, s_t) \tag{DIDO}$$

Under both regimes, the nominal interest rate responds to the dollar-sector inflation. The difference lies in the output gaps that enter the policy rule. In DIDO, the nominal interest rate responds to the dollar-sector output gap.

In Fig. 2, we show the impulse responses of the macroeconomic variables to a 25-basis-point nominal appreciation in the non-dollar currency under the DIAO and DIDO regimes. Both regimes behave similarly when responding to an exchange rate shock, with the impulse response curves of DIDO below those of DIAO, except for the real interest rate. To keep the paper concise, we discuss the impulse responses of DIAO in the main body of the paper and provide the equations of DIDO impulse response functions in Section C.3.

The equation system formed from Eqs. (45), (62), (63) and (DIAO) can be interpreted as one with $\tilde{y}_t$, $\pi_{1,t}$, and $\hat{i}_t$ being the endogenous variables, and s_t being an autoregressive exogenous process. It is then straightforward to express the endogenous variables in terms of the bilateral relative price:

$$\tilde{y}_t = -\lambda\, \upsilon\, \phi_\pi\, \Lambda\, s_t \tag{64}$$

$$\pi_{1,t} = \frac{\upsilon\,(1-\theta)}{\theta}\ (1 - \kappa\, \phi_\pi\, \Lambda)\ s_t \tag{65}$$

$$\hat{i}_t = -\upsilon\ (\kappa - \lambda\, \sigma)\,(1-\theta)\ \phi_\pi\, \Lambda\, s_t \tag{66}$$

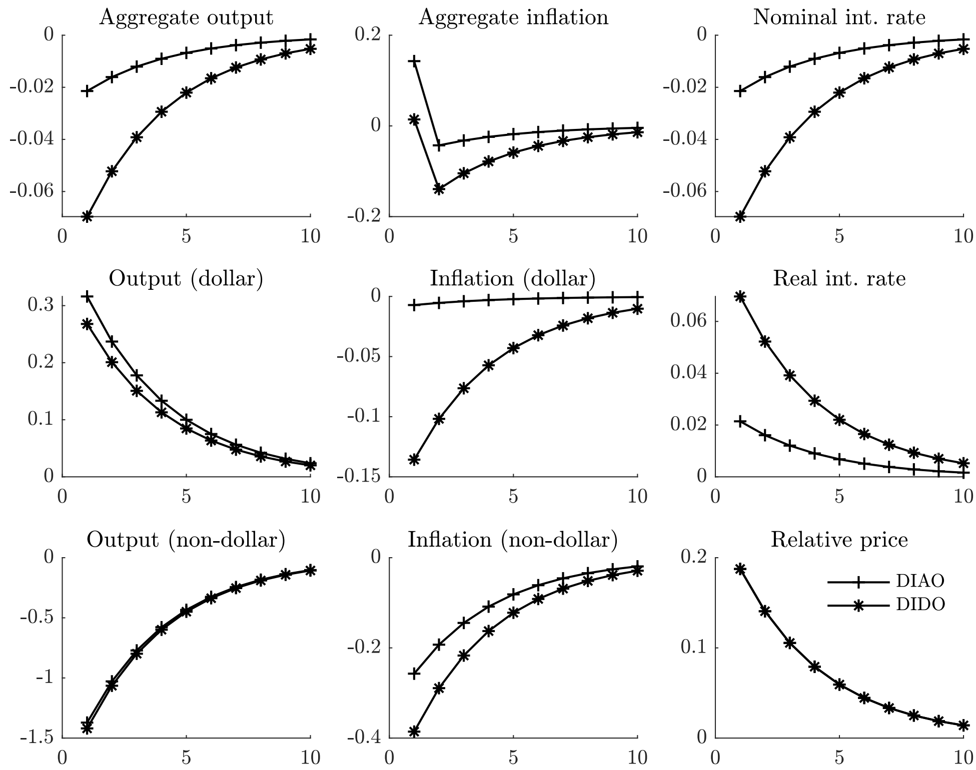

Figure 2: Impulse responses to a 0.25% appreciation of the exchange rate. Under the DIAO regime, the nominal interest rate responds to the dollar-sector inflation and the aggregate output gap. Under the DIDO regime, the nominal interest rate responds to the dollar-sector inflation and the dollar-sector output gap. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange rate shock.

where $\Lambda \equiv \frac{1}{(1-\beta\,\theta)[\sigma\,(1-\theta)+\phi_y]+\kappa\,(\phi_\pi-\theta)} > 0$. Contrary to the baseline case, the one-off exchange rate shock translates into a persistent shock to the bilateral relative price. The variables now behave differently.

Aggregate output in Fig. 2 responds with a level below the natural one and returns to its steady state gradually. The direction of response is the same as in the baseline case shown in Fig. 1. The impulse response decays gradually because the nominal interest rate is a function of the sectoral inflation which returns to its steady state only when the bilateral relative price does. The aggregate output gap, which is sensitive to nominal interest rate changes, also follows the behavior of the bilateral relative price.

The directions of responses of the dollar-sector inflation and the nominal interest rate are no longer unambiguous. They depend on the parameters. In particular, the nominal interest rate decreases when $\kappa\,\sigma^{-1} > \lambda$, also elaborated as

$$\sigma^{-1} > \frac{1-\Theta}{\Theta}\frac{1-\alpha}{\alpha+\varphi} \tag{67}$$

which refers to a sufficiently large elasticity of the output gap to the real interest rate (sufficiently small σ). This condition always holds true in the particular case of a constant return to scale, $\alpha = 0$. The dollar-sector inflation is lower if

$$\frac{(\kappa - \lambda\,\sigma)\;\theta}{1 - \beta\,\theta} > \phi_y \tag{68}$$

which holds true when $\kappa\,\sigma^{-1} > \lambda$, and when either prices are sufficiently rigid (sufficiently large θ), or ϕ_y is sufficiently small. Note that when $\kappa\,\sigma^{-1} < \lambda$, both the nominal interest rate and the dollar-sector inflation increase. From the dynamic IS curve, the real interest rate is found to increase unambiguously, despite the uncertain response of the nominal interest rate:

$$\hat{r}_t = \upsilon\,\lambda\,\sigma\;(1-\theta)\;\phi_\pi\,\Lambda\,s_t > 0. \tag{69}$$

Inflation in the non-dollar sector is derived by adding the inflation differential Eq. (54) to the dollar-sector inflation

$$\pi_{2,t} = -\frac{1-\theta}{\theta}\,(1 - \upsilon + \upsilon\,\kappa\,\phi_\pi\,\Lambda)\;s_t \tag{70}$$

which is always below the steady state. The sectoral output gaps are derived from the demand functions

$$\tilde{y}_{1,t} = -\upsilon\,(\lambda\,\phi_\pi\,\Lambda - \eta)\;s_t \tag{71}$$

$$\tilde{y}_{2,t} = -\left[\lambda\,\upsilon\,\phi_\pi\,\Lambda + \eta\;(1-\upsilon)\right]\;s_t \tag{72}$$

As in the dollar-sector inflation, the response of the dollar-sector output depends on the parameters. For a sufficiently large value of η, firms in the dollar sector produce above the natural level, as households find it easier to substitute one consumption good for another. Firms in the non-dollar sector, instead, always produce below the natural level, as the dollar-denominated price is higher.

Thus far, under the DIAO regime, all the variables presented are only linear in the bilateral relative price. This is due to the choice of a monetary policy rule where the nominal interest rate responds to variables that are associated with the bilateral relative price. However, an exception is aggregate inflation, which is additionally influenced by the contemporaneous exchange rate shock:

$$\pi_t = -\frac{1-\theta}{\theta}\,\upsilon\,\kappa\,\phi_\pi\,\Lambda\,s_t + \upsilon\,\Delta e_t \tag{73}$$

The exchange rate shock offsets some effects of the bilateral relative price. The contemporaneous response of aggregate inflation is always positive, as shown in the appendix. From the second period on, aggregate inflation is below its steady state and is linear only in the bilateral relative price. As a result, a kink is observed in the response of aggregate inflation.

In summary, there is one similarity and two differences between economic dynamics under the AIAO regime and the DIAO (or DIDO) regime. The similarity is that economic dynamics in the aggregate economy and the dollar sector are proportional to the size of the non-dollar sector. For DIAO, this is seen from Eqs. (64), (66) and (73) for the aggregate variables, and from Eqs. (65) and (71) for the dollar-sector variables. In the limiting case where no firm prices in the non-dollar currency, it is expected that the exchange rate shock will not affect the economy.

The two differences between the regimes are as follows. First, the aggregate output gap and inflation experience one-period volatility under the AIAO regime, but they exhibit persistent movements in the DIAO regime. Second, the responses of the nominal interest rate and dollar-sector inflation are unambiguous under the AIAO regime, but they are dependent on the parameter values under the DIAO regime. Dynamics in the non-dollar sector, on the other hand, have shown similar patterns between the two sectors.

6 Equilibrium Dynamics: Heterogeneous Price Rigidity

The implications become richer when price rigidity differs across various pricing currencies. For example, goods listed online may be subjected to more frequent price changes as it is less costly for merchants to do so online than at physical boutiques. In the future, perhaps large online merchants will introduce their own cryptocurrency and price their goods in that currency, just like the airlines price the award tickets in miles, giving rise to the considerations here. In this section, we thus analyze the economic dynamics when the dollar and the non-dollar sectors differ in the extent of price rigidity.

The NK framework with different price rigidities and the baseline AIAO monetary policy are summarized by the following five equations:

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1} \left[\hat{i}_t - \mathrm{E}_t\left[(1-\upsilon)\,\pi_{1,t+1} + \upsilon\,\pi_{2,t+1}\right] - \hat{r}_t^n\right] \tag{74}$$

$$\pi_{1,t} = \beta\,\mathrm{E}_t[\pi_{1,t+1}] + \kappa_1\,\tilde{y}_t + \lambda_1\,\upsilon\,s_t \tag{75}$$

$$\pi_{2,t} = \beta\,\mathrm{E}_t[\pi_{2,t+1}] + \kappa_2\,\tilde{y}_t - \lambda_2\,(1-\upsilon)\,s_t \tag{76}$$

$$s_t = s_{t-1} + \pi_{2,t} - \pi_{1,t} + \Delta e_t \tag{77}$$

$$\hat{i}_t = \phi_\pi\,\left[(1-\upsilon)\;\pi_{1,t} + \upsilon\;(\pi_{2,t} + \Delta e_t)\right] + \phi_y\,\tilde{y}_t \tag{78}$$

Notice that the parameters κ_1, κ_2, λ_1, and λ_2 are now shown with subscripts as the price rigidities differ. This equation system contains the aggregate output gap, the two sectoral inflation rates, and the nominal interest rate as the endogenous variables, the bilateral relative price as the state variable, and the nominal appreciation of the non-dollar currency as the exogenous variable. Since the price rigidities are different between the two sectors, Proposition 2 no longer holds, meaning aggregate inflation is not necessarily independent of the bilateral relative price as in our earlier simulations of the AIAO regime.

We first consider a scenario in which the non-dollar sector has a lower price rigidity than the dollar sector. This is done by adjusting the parameter θ_2, so that it corresponds to two price changes per year ($\theta_2 = 0.5$) and four price changes per year ($\theta_2 = 0$), while holding the frequency of price changes in the dollar sector at once a year ($\theta_1 = 0.75$). The impulse responses are shown in Fig. 3. The circle-marked lines are the impulse responses in Fig. 1, when the frequency of price change in the non-dollar sector is the same as that in the dollar sector. As price rigidity is reduced in the non-dollar sector, more firms respond to the exchange rate shock by adjusting their non-dollar prices, offsetting the increase in their dollar-denominated prices. As a result, we see a smaller impact of the exchange rate shock on the bilateral relative price. This leads to a smaller substitution effect between the consumption goods, and hence smaller responses in most of the variables. Also note that aggregate output, aggregate inflation, and the nominal interest rate no longer experience one-off volatility. They take time to return to the steady states. In the limiting case where all firms in the non-dollar sector can adjust prices in response to the exchange rate shock ($\theta_2 = 0$), the change in inflation in the non-dollar sector fully offsets the impact of the exchange rate shock, leading to no change in their dollar-denominated prices. Therefore, when prices are fully flexible in the non-dollar sector, the exchange rate shock is neutral, and does not influence economic dynamics. This result is in line with Proposition 4.

We then examine the economic dynamics when the dollar sector becomes more flexible, as compared to the non-dollar sector. Similar to the above simulation, we vary the price rigidity in the dollar sector while holding the frequency of price change in the non-dollar

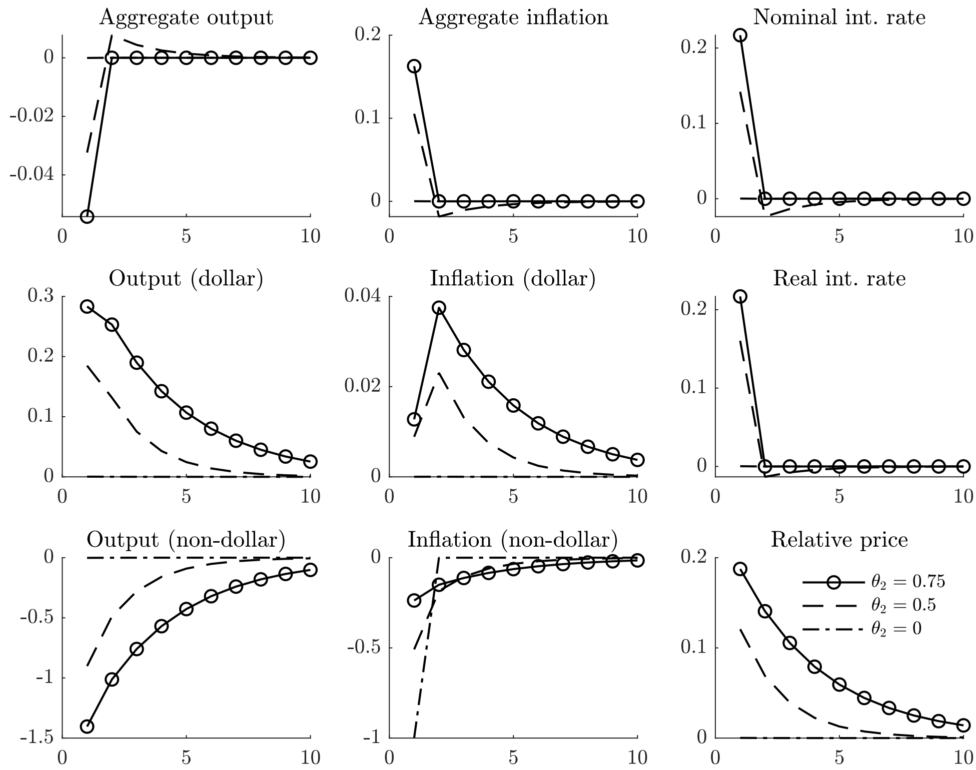

Figure 3: Impulse responses to a 0.25% appreciation of the exchange rate. Price rigidity in the dollar sector is kept at $\theta_1 = 0.75$, corresponding to a frequency of one price change per year. Price rigidity in the non-dollar sector varies among 0.75, 0.5, and 0, corresponding to one, two and four price changes per year, respectively. The monetary policy regime is AIAO. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange rate shock.

sector at once a year. Fig. 4 shows the impulse responses when the price rigidity in the dollar sector changes from one change a year to two and four changes a year. When the non-dollar sector is hit by an exchange rate shock, the rigidity in the non-dollar sector causes the dollar-denominated price of its goods to be higher. The re-distributed demand from the non-dollar sector to the dollar sector motivates the firms in the dollar sector to price higher. When price rigidity in the dollar sector is lower, more firms increase their prices to meet the increased demand. We therefore observe that the response of the dollar-sector inflation first shows a dip, caused by the one-off exchange rate shock, then a value in the second period that increases with price flexibility.

It is important to note that even when prices are fully flexible in the dollar sector, the exchange rate shock is not neutral as in the case of the flexible non-dollar price. Again, this is in line with Proposition 4. The non-neutrality arises because the dollar-denominated price of the non-dollar sector goods resulting from the exchange rate shock does not coincide with the desired price level. As a result, when the dollar-sector firms price their goods at the desired levels, there is a price differential between the goods from

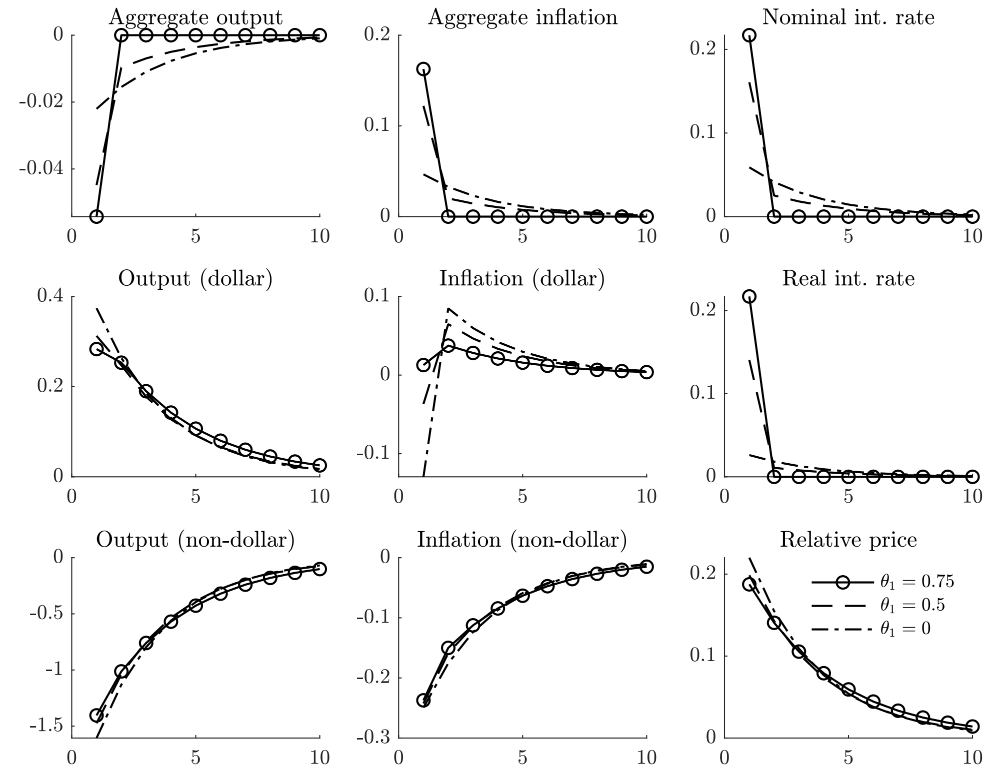

Figure 4: Impulse responses to a 0.25% appreciation of the exchange rate. Price rigidity in the non-dollar sector is kept at $\theta_2 = 0.75$, corresponding to a frequency of one price change per year. Price rigidity in the dollar sector varies among 0.75, 0.5, and 0, corresponding to one, two and four price changes per year, respectively. The monetary policy regime is AIAO. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange rate shock.

the two sectors. While firms in the non-dollar sector take time to reset prices to their desired level, firms in the dollar sector optimize their prices every period. Therefore, we see the non-negligible impulse responses even when prices are fully flexible in the dollar sector.

To obtain a more general picture of economic dynamics at different extents of price rigidity, in Fig. 5, we compute the cumulative impulse responses for the aggregate output gap and aggregate inflation over two years (eight quarters). As expected, when prices are flexible in the non-dollar sector ($\theta_2 = 0$), an exchange rate shock does not cause any movements in the aggregate output gap and aggregate inflation. However, when prices are also flexible in the dollar sector ($\theta_1 = 0$), the exchange rate shock is not neutral, unless prices are flexible in the non-dollar sector. Furthermore, throughout the range of dollar-sector price rigidity, the cumulative impulse responses are generally greater when price rigidity in the non-dollar sector is higher.

In the previous section, we have shown that the impulse responses to an exchange rate shock for the overall economy and the dollar sector are proportional to the size of

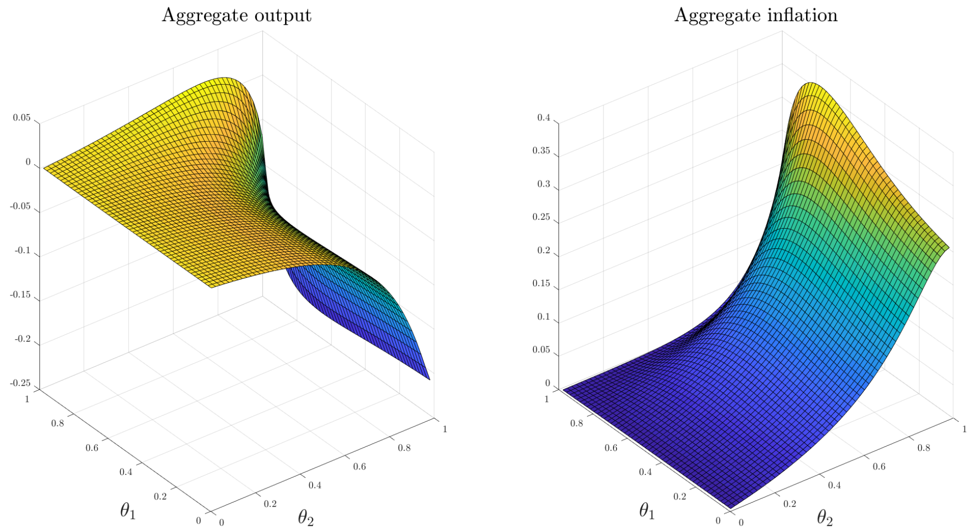

Figure 5: Two-year cumulative responses of the aggregate output gap and aggregate inflation to a 0.25% appreciation of the exchange rate. The vertical axes indicate cumulative percentage deviations from the steady states.

the non-dollar sector. To see if this result also holds in an environment of heterogeneous price rigidities, we simulate for different values of υ. The impulse responses are shown in Fig. 6. The price rigidities are set to $\theta_1 = 0.75$ for the dollar sector, and $\theta_2 = 0.5$ for the non-dollar sector. When the size of the non-dollar sector increases from 0.2 to 0.5, the responses of the economy increase accordingly, whereas when the size of the non-dollar sector diminishes to 0, there are no responses to the exchange rate shock. Therefore, we infer that, with heterogeneous price rigidities, the responses of the overall economy and the dollar sector to an exchange rate shock also vary with the size of the non-dollar sector.

7 Currency Choice

We now depart from the assumption that a firm sticks to the same pricing currency all the time. We allow firms to choose a different pricing currency when they are allowed to reset prices. As a result, a rational firm chooses the currency that provides it with the best utility outcome.

We introduce a discrete choice model building on McFadden (1974) to specify the conditions under which firms choose to price in one currency versus the other. Let $\mathcal{U}_{i,j,t}$ denote the utility of producer i, who resets its price in period t and uses currency j to do so. We specify

$$\log\left(\mathcal{U}_{i,j,t}\right) = \log\left(\mathcal{V}_{j,t,t}\right) + \varepsilon_{i,j,t} \tag{79}$$

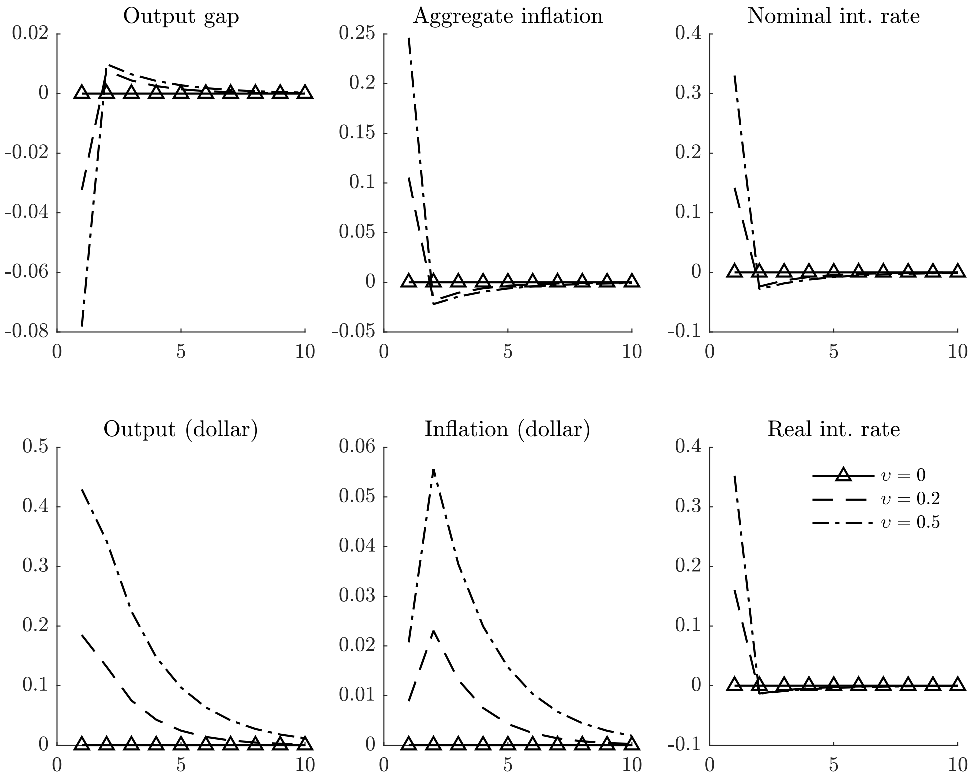

Figure 6: Impulse responses to a 0.25% appreciation of the exchange rate. Price rigidity prarameters are set to $\theta_1 = 0.75$ and $\theta_2 = 0.5$. The monetary policy regime is AIAO. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange rate shock.

where $\mathcal{V}_{j,s,t}$ is the continuation value to the firm as the sum of expected and discounted optimized profits when the price has been reset at $s \leq t$, using currency j to do so, and where $\varepsilon_{j,t}$ is an exogenous and random preference parameter for picking currency j at time t. One may think of this parameter as arising from regulatory concerns, customer relationships or network effects for using a particular currency, which are outside this model. We assume $\varepsilon_{j,t}$ to be independent across i, j and t as well as any other shocks and to be distributed according to a type I extreme value distribution

$$F\left(\varepsilon_{i,j,t}\right) = \exp\left(-\gamma_j \exp(-\varepsilon_{i,j,t})\right) \tag{80}$$

where $\gamma_j > 0$ is a constant and common across all producers and dates. A firm chooses to price in currency j if the utility from the optimized value is at least as high as that from all the other alternatives:

$$\mathcal{U}_{j,t} \geq \mathcal{U}_{j',t} \ \forall j' = 0, ..., K. \tag{81}$$

which holds when

$$\varepsilon_{j',t} \leq \log\left(\frac{\mathcal{V}_{j,t,t}}{\mathcal{V}_{j',t,t}}\right) + \varepsilon_{j,t} \ \forall j' = 1, ..., J. \tag{82}$$

The joint probability from J currencies gives the probability of a firm pricing in currency j:

$$\mathrm{Pr}_{j,t} = \frac{\gamma_j\, \mathcal{V}_{j,t,t}}{\sum_{j'=1}^{J} \gamma_{j'}\, \mathcal{V}_{j',t,t}} \tag{83}$$

We write the value to the firm in the following recursive form

$$\mathcal{V}_{j,s,t} = \Xi\left(p^*_{s,j} + e_{j,t} \,|\, x_t\right) + \theta_j \mathrm{E}_t Q_{t,t+1} \mathcal{V}_{j,s,t+1} \tag{84}$$

where $\Xi\,(p\,|\,x_t)$ is the profit at the log dollar price p given the state of the economy x_t, and where $p^*_{s,j}$ is the log of the optimal currency-j price set in period s. In the period following the price adjustment, the firm continues to use the current optimal price with probability θ_j. Otherwise, a price-setting opportunity arises. For ease of exposition, we assume that firms are "reborn" with a different owner. Thus, the continuation value for the old firm owner, in the case of a price-setting opportunity, is zero.[2] A proposition from the baseline case of homogeneous price rigidity can be summarized as follows, echoing a result in Gopinath et al. (2010):

Proposition 5. *Assume homogeneous price rigidity in a two-sector economy. Let $\gamma_1 = 1$, $\gamma_2 = \gamma$. The second-order approximation to the probability of pricing in dollars is*

$$\mathrm{Pr}_{1,t} \approx \frac{1}{1+\gamma} + \frac{\gamma\, K\,(x_t)}{(1+\gamma)^2\, \tilde{\mathcal{V}}_t} \sum_{\ell=0}^{\infty} (\beta\theta)^{\ell}\, \mathrm{Var}_t\,(e_{t+\ell}) \left[\frac{1}{2} - \frac{\mathrm{Cov}_t\,(\tilde{p}_{t+\ell}, e_{t+\ell})}{\mathrm{Var}_t\,(e_{t+\ell})}\right] \tag{85}$$

where $K(x_t) \equiv -\tilde{\Xi}_{pp}(\tilde{p}_t \mid x_t)$ and $\tilde{\mathcal{V}}_t$ is the value function of a firm always using the optimal flexible dollar price, both evaluated at the date-t optimal log flex price $\tilde{p}_t$.

Proof. See Section A.5. $\square$

The endogenous currency choice leads to time-varying sizes of the currency sectors. With a time subscript to the parameter υ_j, the law of motion for $\upsilon_{j,t}$ is:

$$\upsilon_{j,t} = \theta_j\, \upsilon_{j,t-1} + \mathrm{Pr}_{j,t} \sum_{j'=1}^{K} (1-\theta_{j'})\ \upsilon_{j',t-1} \tag{86}$$

[2] Alternatively, one could assume that it is the same firm owner, giving rise to an additional additive term $(1-\theta_j)\,\mathrm{E}_t Q_{t,t+1} \mathcal{U}_t$, where the continuation value $\mathcal{U}_t = E_t[\max_j \mathcal{U}_{i,j,t}]$ is the maximized producer utility, taking expectations over $\epsilon_{i,j,t}$.

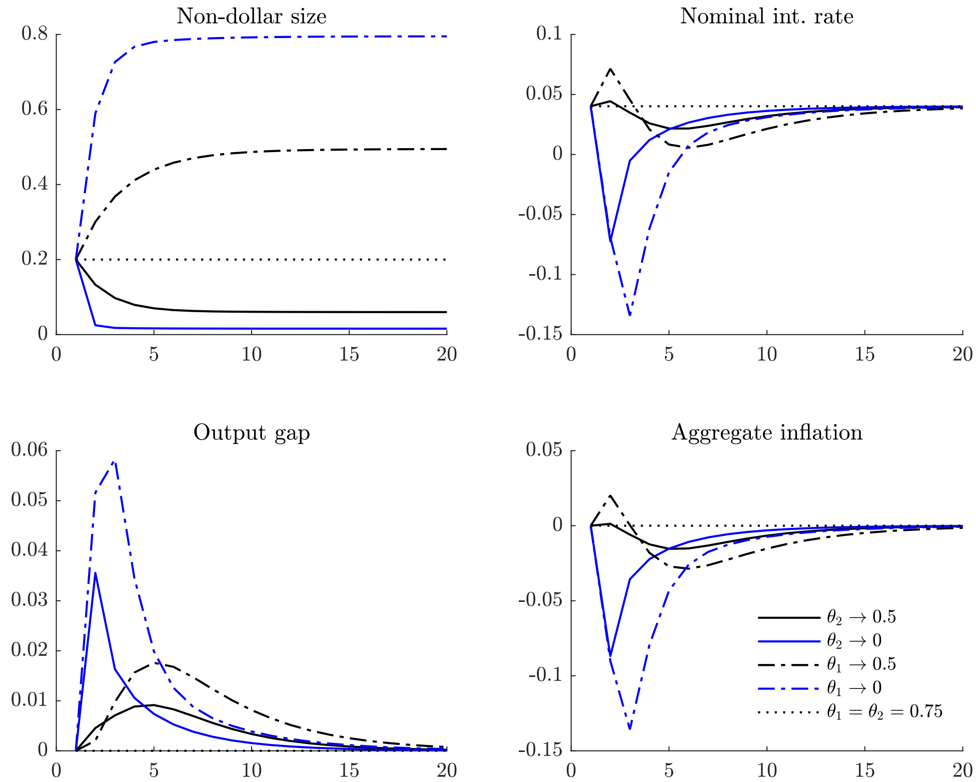

Figure 7: Transition dynamics to more flexible prices.

The first term is the proportion of firms that did not change prices from the previous period. The second term arises as the sum over all firms that get to reset their price as well as choose their currency, and choose currency j.

A more general case pertains to the transition dynamics as a currency sector experiences a permanent change in price rigidity. Extending Proposition 5, we examine the transition dynamics from the baseline specification to different extents of price rigidities in a two-sector economy. We use Dynare and the system of non-linear equilibrium conditions for the numerical calculations. The results are shown in Fig. 7. The non-dollar sector increases in size when price rigidity in the dollar sector is lower. This is because, with lower price rigidity, firms in the dollar sector switch to pricing in the non-dollar currency more easily. For the same reason the non-dollar sector decreases in size when price rigidity in the non-dollar sector is lower. In general, when transiting from the baseline specification to one with lower price rigidity, the steady states of the key macroeconomic variables are unchanged. However, the variables deviate from the steady states temporarily.

The evolving size of the currency sector is related to our earlier finding of proportional macroeconomic responses to the size of the non-dollar sector under an exchange rate shock. In an environment with endogenous currency choice, the central bank may see changes in the impact of exchange rate shocks when price rigidity in the dollar or the non-dollar sector changes. In particular, when prices in the dollar sector become less rigid, exchange rate shocks may pose an increasing risk to economic stability when the non-dollar sector grows in size.

When price rigidity stays constant, so will the pricing choice of firms. In that case, the log-linearized analysis of the previous sections applies even when allowing for the endogeneity of the currency choice.

8 Conclusion

Extending the basic NK framework in Galí (2015), we examine the macroeconomic dynamics when multiple parallel currencies co-exist in an economy. Our baseline case with homogeneous price rigidity finds an NKPC that is similar to a conventional one, with an additional disturbance term from the exchange rate. Upon relaxing the assumption of homogeneous price rigidity, we find that the exchange rate shock is neutral only when the shock originates from a currency sector with flexible prices. We have also discussed a scenario in which firms can change the pricing currency. We find that the non-dollar sector may increase in size when prices in the dollar sector are less rigid, posing a higher risk to economic stability.

Our analyses are only forward-looking and not meant to replicate existing situations. The idea that a considerable proportion of firms price in a non-official currency may seem futuristic. Existing regulatory frameworks are mostly based on the dollar, resulting in the majority of firms using the dollar as their pricing unit. However, private monies such as Diem and stablecoins have made the headlines. At the same time, countries are contemplating cross-border retail payment facilities. There is a possibility that official and private currencies can both be pricing units.

The framework presented in this paper is at most a stereotype model. A benefit of this simple model is that it is easy to build on it to analyze more complicated issues, for example, a non-random path of the exchange rate, which we shall leave for future research.

References

Adrian, T. and T. Mancini-Griffoli (2019). The rise of digital money. Fin Tech Notes, IMF.

Asimakopoulos, S., M. Lorusso, and F. Ravazzolo (2019). A new economic framework: A DSGE model with cryptocurrency. draft, Free University of Bozen-Bolzano.

Barrdear, J. and M. Kumhof (2016). The macroeconomics of central bank issued digital currencies. Bank of England working paper no. 605.

Barsky, R. B., C. L. House, and M. S. Kimball (2007, June). Sticky-Price Models and Durable Goods. American Economic Review 97(3), 984–998.

Baughman, G. and J. Flemming (2020). Equilibrium bitcoin pricing. Finance and Economics Discussion Series 2020-048, Board of Governors of the Federal Reserve System, Washington.

Benigno, P. (2019). Monetary policy in a world of cryptocurrencies. CEPR Discussion Paper No. DP13517.

Benigno, P., L. Schilling, and H. Uhlig (2019). Cryptocurrencies, currency competition, and the impossible trinity. Becker Friedman Institute for Economics Working Paper 2019-108.

Berentsen, A. (1998). Monetary implications of digital money. Kyklos 51, 89–117.

Biais, B., C. Bisière, M. Bouvard, C. Casamatta, and A. J. Menkveld (2020). Equilibrium bitcoin pricing. EconPol Working Paper (45).

Brunnermeier, M. K., H. James, and J.-P. Landau (2019). The digitalization of money. draft, Princeton University.

Brunnermeier, M. K. and D. Niepelt (2019). On the equivalence of private and public money. Technical report, National Bureau of Economic Research.

Calvo, G. A. (1983, September). Staggered prices in a utility-maximizing framework. Journal of Monetary Economics 12(3), 383–398.

Chiu, J. and T. V. Koeppl (2019). The economics of cryptocurrencies—bitcoin and beyond. Staff Working Paper, Document de travail du personnel 2019-40, Bank of Canada, Banque Du Canada.

Cienfuegos, N. C. (2019, January). The Importance of Production Networks and Sectoral Heterogeneity for Monetary Policy. Technical report. University of Chicago.

Duffie, D. (2019). Digital currencies and fast payment systems: Disruption is coming. draft, Stanford University.

Fernández-Villaverde, J. and D. Sanches (2019, October). Can currency competition work? Journal of Monetary Economics 106, 1–15.

Ferrari, M. M., A. Mehl, and L. Stracca (2020, November). Central bank digital currency in an open economy. ECB Working Paper no. 2488.

Galí, J. (2015, June). Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications - Second Edition. Princeton University Press.

Garratt, R. and N. Wallace (2018). Bitcoin 1, bitcoin 2,....: An experiment in privately issued outside monies. Economic Inquiry 56(3), 1887–1897.

George, A., T. Xie, and J. D. Alba (2020). Central bank digital currency with adjustable interest rate in small open economies. Available at SSRN 3605918.

Girton, L. and D. Roper (1981). Theory and implications of currency substitution. Journal of Money, Credit and Banking 13(1), 12–30.

Gopinath, G., E. Box, C. Casas, F. J. Díez, P.-O. Gourinchas, and M. Plagborg-Møller (2020). Dominant currency paradigm. American Economic Review 110(3), 677–719.

Gopinath, G., O. Itskhoki, and R. Rigobon (2010, March). Currency Choice and Exchange Rate Pass-Through. American Economic Review 100(1), 304–336.

Grossa, J. and J. Schillerb (2020). A model for central bank digital currencies: Do CBDCs disrupt the financial sector? Available at SSRN.

Kareken, J. and N. Wallace (1981). On the indeterminacy of equilibrium exchange rates. The Quarterly Journal of Economics 96(2), 207–222.

Manuelli, R. E. and J. Peck (1990). Exchange rate volatility in an equilibrium asset pricing model. International Economic Review, 559–574.

Matsuyama, K., N. Kiyotaki, and A. Matsui (1993). Toward a theory of international currency. The Review of Economic Studies 60(2), 283–307.

McFadden, D. (1974). Conditional logit analysis of qualitative choice behaviour. In P. Zarembka (Ed.), Frontiers in Econometrics, pp. 105–142. New York, NY, USA: Academic Press New York.

Pasten, E., R. Schoenle, and M. Weber (2018). The propagation of monetary policy shocks in a heterogeneous production economy. Becker Friedman Institute for Economics Working Paper (2018-78).

Rubbo, E. (2020). Networks, Phillips curves, and monetary policy. draft, Princeton University.

Schär, F. and A. Berentsen (2020). Bitcoin, Blockchain, and Cryptoassets: A Comprehensive Introduction. MIT Press, Boston, MA.

Schilling, L. and H. Uhlig (2019a). Currency substitution under transaction costs. Available at SSRN 3314249.

Schilling, L. and H. Uhlig (2019b, October). Some simple bitcoin economics. Journal of Monetary Economics 106, 16–26.

Sterk, V. (2010, March). Credit frictions and the comovement between durable and non-durable consumption. Journal of Monetary Economics 57(2), 217–225.

Appendix A Proofs of propositions

A.1 Proof of Proposition 1

The optimality conditions for real holdings of any currency k is

$$\xi\, l_t - \sigma\, c_t = \frac{\beta}{1-\beta}\left[\sigma\,(c_t - \mathrm{E}_t[c_{t+1}]) - \mathrm{E}_t[\pi_{t+1}] + \mathrm{E}_t[\Delta e_{k,t+1}] - (1-\rho_z)\, z_t\right] \tag{A.1}$$

Taking the difference between $k = j$ and $k = j'$ proves the proposition

$$e_{j,t} - e_{j',t} = \mathrm{E}_t\left[e_{j,t+1} - e_{j',t+1}\right]. \tag{A.2}$$

A.2 Proof of Proposition 2

First result With the exchange rate being a random-walk process, the optimal price for any currency sector k is rewritten as:

$$p^*_{k,t} + e_{k,t} = (1-\beta\,\theta_k)\sum_{h=0}^{\infty}(\beta\theta_k)^h\,\mathrm{E}_t\left[\tilde{p}_{t+h}\right] \tag{A.3}$$

Let the price rigidity be the same for an arbitrary pair of currencies j and j', $\theta_j = \theta_{j'} = \theta$. Eq. (A.3) can be written as $p^*_{k,t} + e_{k,t} = (1-\beta\,\theta)\sum_{h=0}^{\infty}(\beta\theta)^h\,\mathrm{E}_t\left[\tilde{p}_{t+h}\right]$, in which the right-hand sides are identical, and independent of the currency choice. Hence, prices set in currencies j and j' are equivalent. Mathematically, $p^*_{j,t} + e_{j,t} = p^*_{j',t} + e_{j',t}$.

Second result Price levels in sectors j and j' are weighted sums of optimal prices and the price levels from the previous period: $p_{j,t} = (1-\theta)\,p^*_{j,t} + \theta\,p_{j,t-1}$ and $p_{j',t} = (1-\theta)\,p^*_{j',t} + \theta\,p_{j',t-1}$. From the definition of $s_{jj',t}$:

$$\begin{aligned}
s_{jj',t} &= p_{j,t} + e_{j,t} - (p_{j',t} + e_{j',t})\\
&= (1-\theta)\left(p^*_{j,t} - p^*_{j',t}\right) + \theta\,\left(p_{j,t-1} - p_{j',t-1}\right) + e_{j,t} - e_{j',t}\\
&= (1-\theta)\left(p^*_{j,t} + e_{j,t} - p^*_{j',t} - e_{j',t}\right) + \theta\,\left(p_{j,t-1} + e_{j,t} - p_{j',t-1} - e_{j',t}\right)\\
&= \theta\, s_{jj',t-1} + \theta\,\left(\Delta e_{j,t} - \Delta e_{j',t}\right)
\end{aligned}$$

The last equation makes use of the first result.

Third and fourth results The expression for $s_{jj',t}$ from the second result can be rearranged as an expression for s_{t-1}: $s_{jj',t-1} = \frac{1}{\theta}\, s_{jj',t} - \Delta e_{j,t} + \Delta e_{j',t}$. The law of motion of $s_{jj',t} = s_{jj',t-1} + \pi_{j,t} + \Delta e_{j,t} - \pi_{j',t} - \Delta e_{j',t}$ is rearranged as

$$\pi_{j,t} - \pi_{j',t} = s_{jj',t} - s_{jj',t-1} - \Delta e_{j,t} + \Delta e_{j',t} = -\frac{1-\theta}{\theta}\, s_{jj',t}$$

For the output gap differential:

$$\tilde{y}_{j,t} - \tilde{y}_{j',t} = -\eta\, \hat{p}_{j,t} + \tilde{y}_t - (-\eta\, \hat{p}_{j',t} + \tilde{y}_t) = -\eta\, s_{jj',t}$$

A.3 Proof of Proposition 3

Let $\theta_j = \theta$ for all j. The second and third terms on the right-hand side of Eq. (43) are simplified as $\boldsymbol{\upsilon}'\boldsymbol{\kappa}\,\tilde{y}_t = \kappa\,\tilde{y}_t$ and $-\lambda\boldsymbol{\upsilon}'\hat{\boldsymbol{p}}_t = 0$, where $\kappa \equiv \Theta\,\frac{(1-\theta)(1-\beta\,\theta)}{\theta}\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)$. Consequently, the NKPC for aggregate inflation is expressed independent of the relative price.

A.4 Proof of Proposition 4

From Eq. (22) and equilibrium conditions, the relative price in a sector j with flexible prices is a function of the output gap

$$\hat{p}_{j,t} = p_{j,t} + e_{j,t} - p_t = \Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)\tilde{y}_t \tag{A.4}$$

The contemporaneous sectoral inflation is

$$\pi_{j,t} = \Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)\Delta\tilde{y}_t + \pi_t - \Delta e_{j,t} \tag{A.5}$$

The contemporaneous aggregate inflation is

$$\pi_t = \upsilon_j \pi_{j,t} + \sum_{k\neq j} \upsilon_k\left(\pi_{k,t} + \Delta e_{k,t}\right) = \zeta_{\upsilon_j}\Delta\tilde{y}_t + \sum_{k\neq j}\frac{\upsilon_k}{1-\upsilon_j}\left(\pi_{k,t} + \Delta e_{k,t}\right) \tag{A.6}$$

where $\zeta_{\upsilon_j} \equiv \frac{\upsilon_j\Theta}{1-\upsilon_j}\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)$. Take expectations on both sides of Eq. (A.5). The expected sectoral inflation is

$$\mathrm{E}_t[\pi_{j,t+1}] = \Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)\left(\mathrm{E}_t[\tilde{y}_{t+1}] - \tilde{y}_t\right) + \mathrm{E}_t[\pi_{t+1}] \tag{A.7}$$

Take expectations on both sides of Eq. (A.6). The expected aggregate inflation is

$$\mathrm{E}_t[\pi_{t+1}] = \zeta_{\upsilon_j}\,\mathrm{E}_t[\Delta\tilde{y}_{t+1}] + \sum_{k\neq j}\frac{\upsilon_k}{1-\upsilon_j}\,\mathrm{E}_t[\pi_{k,t+1}] \tag{A.8}$$

Substitute the expected inflation in the IS curve with Eq. (A.8):

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] + \zeta_{\upsilon_j}\,\sigma^{-1}\left(\mathrm{E}_t[\tilde{y}_{t+1}] - \tilde{y}_t\right) - \sigma^{-1}\left(\hat{i}_t - \sum_{k\neq j}\frac{\upsilon_k}{1-\upsilon_j}\,\mathrm{E}_t[\pi_{k,t+1}]\right) \tag{A.9}$$

$$= \mathrm{E}_t[\tilde{y}_{t+1}] - \left(\sigma + \zeta_{\upsilon_j}\right)^{-1}\left(\hat{i}_t - \sum_{k\neq j}\frac{\upsilon_k}{1-\upsilon_j}\,\mathrm{E}_t[\pi_{k,t+1}]\right) \tag{A.10}$$

The NK framework with sector j being the flexible one has the following $2K$ equations, without the NKPC for sector j. The subscript $-j$ indicate vectors without the jth element.

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \left(\sigma + \zeta_{\upsilon_j}\right)^{-1}\left(\hat{i}_t - \frac{\boldsymbol{\upsilon}'_{-j}\boldsymbol{\pi}_{-j,t+1}}{1-\upsilon_j}\right) \tag{A.11}$$

$$\boldsymbol{\pi}_{-j,t} = \beta\,\mathrm{E}_t\left[\boldsymbol{\pi}_{-j,t+1}\right] + \boldsymbol{\kappa}_{-j}\,\tilde{y}_t - \boldsymbol{\lambda}_{-j}\circ\hat{\boldsymbol{p}}_{-j,t} \tag{A.12}$$

$$\hat{\boldsymbol{p}}_{-j,t} = \hat{\boldsymbol{p}}_{-j,t-1} + \left(\mathbf{I} - \frac{\mathbf{1}\,\boldsymbol{\upsilon}'_{-j}}{1-\upsilon_j}\right)\left(\boldsymbol{\pi}_{-j,t} + \Delta\boldsymbol{e}_{-j,t}\right) - \zeta_{\upsilon_j}\Delta\tilde{y}_t\mathbf{1} \tag{A.13}$$

$$\hat{i}_t = \phi_\pi\,\pi_t + \phi_y\,\tilde{y}_t \tag{A.14}$$

where $\pi_t = \zeta_{\upsilon_j}\Delta\tilde{y}_t + \frac{\boldsymbol{\upsilon}'_{-j}(\boldsymbol{\pi}_{-j,t}+\Delta\boldsymbol{e}_{-j,t})}{1-\upsilon_j}$. The exchange rate shock from sector j, $\Delta e_{j,t}$, does not enter the equation system, so it does not lead to changes in aggregate inflation and output gap. However, any shock from another sticky sector leads to changes in the aggregate output gap. According to Eq. (A.4), the price level in sector j changes only when there are changes in the aggregate output gap.

A.5 Proof of Proposition 5

The proof and its logic are patterned after Gopinath et al. (2010). For a two-sector economy, let $\gamma_1 = 1$, and $\gamma_2 = \gamma$. The probability of pricing in dollars is $\mathrm{Pr}_t = \frac{\mathcal{V}_{1,t,t}}{\mathcal{V}_{1,t,t}+\gamma\,\mathcal{V}_{2,t,t}}$. A first-order approximation around the flexible-price optimum $\mathcal{V}_{1,t,t} = \mathcal{V}_{2,t,t} = \tilde{\mathcal{V}}_t$ delivers

$$\mathrm{Pr}_t \;\approx\; \widetilde{\mathrm{Pr}} + \frac{\partial \mathrm{Pr}_t}{\partial \mathcal{V}_{1,t,t}}_{|\mathcal{V}_{1,t,t}=\mathcal{V}_{2,t,t}=\tilde{\mathcal{V}}_t}\left(\mathcal{V}_{1,t,t} - \tilde{\mathcal{V}}_t\right) + \frac{\partial \mathrm{Pr}_t}{\partial \mathcal{V}_{2,t,t}}_{|\mathcal{V}_{1,t,t}=\mathcal{V}_{2,t,t}=\tilde{\mathcal{V}}_t}\left(\mathcal{V}_{2,t,t} - \tilde{\mathcal{V}}_t\right) \tag{A.15}$$

$$= \frac{1}{1+\gamma} + \frac{\gamma}{(1+\gamma)^2\tilde{\mathcal{V}}_t}\left(\mathcal{V}_{1,t,t} - \mathcal{V}_{2,t,t}\right) \tag{A.16}$$

"Telescope out," i.e., iterate on (84) to obtain[3]

$$\mathcal{V}_{2,t,t} = \max_{p^*_{t,2}} E_t\left[\sum_{l=0}^{\infty}\theta^l Q_{t,t+l}\Xi\left(p^*_{s,2} + e_{2,t+l}\,|\,x_{t+l}\right)\right] \tag{A.17}$$

[3]There is only one exchange rate, which is the exchange rate of the second currency with respect to the dollar. For notational consistency, we keep the subindex 2 on that exchange rate.

Consider the second-order approximation of the profit function around the log of the flex-price optimum, i.e.

$$\Xi(p\,|\,x_{t+l}) \approx \tilde{\Xi}_{t+l} + \frac{1}{2}\Xi''_{t+l}(p - \tilde{p}_{t+l})^2 \tag{A.18}$$

where $\tilde{\Xi}_{t+l} = \Xi(\tilde{p}_{t+l}\,|\,x_{t+l})$ is the profit at the optimal flexible price in $t+l$ and where Ξ''_{t+l} is the second derivative of $\Xi(\cdot\,|\,x_{t+l})$ at $\tilde{p}_{t+l}$. Note that the first derivative is zero by virtue of the optimality of $\tilde{p}_{t+l}$. Replace the profit function in (A.17) with the second-order approximation in (A.18). Additionally assume that $\Xi''_{t+l} = \Xi''_t + \epsilon_{t+1}$, where ϵ_{t+l} is (approximately) independent of other sources of randomness. Likewise, assume that $Q_{t,t+l} = \beta^l$ up to a term (approximately) independent of other sources of randomness. We obtain

$$\mathcal{V}_{2,t,t} = \max_{p^*_{t,2}} E_t\left[\sum_{l=0}^{\infty}(\beta\theta)^l\left(\tilde{\Xi}_{t+l} + \frac{1}{2}\Xi''_t(p^*_{t,2} + e_{2,t+l} - \tilde{p}_{t+l})^2\right)\right] \tag{A.19}$$

The first-order condition

$$0 = E_t\left[\sum_{l=0}^{\infty}(\beta\theta)^l\Xi''_t(p^*_{t,2} + e_{2,t+l} - \tilde{p}_{t+l})\right] \tag{A.20}$$

can be simplified to

$$p^*_{t,2} = E_t[\tilde{p}_{t+l} - e_{2,t+l}] \tag{A.21}$$

when we assume that the right-hand side is (approximately) independent of l: we shall do so. With that (A.19) becomes

$$\mathcal{V}_{2,t,t} = E_t\left[\sum_{l=0}^{\infty}(\beta\theta)^l\tilde{\Xi}_{t+l}\right] + \Xi''_t\sum_{l=0}^{\infty}(\beta\theta)^l\left(\frac{1}{2}\mathrm{Var}_t[e_{2,t+l}] - \mathrm{Cov}(e_{2,t+l},\tilde{p}_{t+l}) + \frac{1}{2}\mathrm{Var}_t[\tilde{p}_{t+l}]\right) \tag{A.22}$$

For the dollar, the same calculation delivers a simpler expression, since there are no exchange rate terms,

$$\mathcal{V}_{1,t,t} = E_t\left[\sum_{l=0}^{\infty}(\beta\theta)^l\tilde{\Xi}_{t+l}\right] + \Xi''_t\sum_{l=0}^{\infty}(\beta\theta)^l\frac{1}{2}\mathrm{Var}_t[\tilde{p}_{t+l}] \tag{A.23}$$

Plug (A.22) and (A.23) into (A.16) and rewrite to obtain equation (85).

TECHNICAL APPENDIX (FOR ONLINE PUBLICATION)

Appendix B Mathematical Derivations for the Linearized NK Model

B.1 Households

The lifetime utility of a representative household is given by:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, L_t, N_t\right) Z_t \tag{B.1}$$

$$u\left(C_t, L_t, N_t\right) = \frac{C_t^{1-\sigma}-1}{1-\sigma} + \frac{L_t^{1-\xi}-1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi} \tag{B.2}$$

The marginal utility of consumption, liquidity, and labor:

$$u_{C,t} = C_t^{-\sigma} \tag{B.3}$$

$$u_{L,t} = L_t^{-\xi} \tag{B.4}$$

$$u_{N,t} = -N_t^{\varphi}. \tag{B.5}$$

The household's budget constraint is

$$C_t + B_t + \sum_{j=1}^{J} L_{j,t} = \frac{\exp\left(i_{t-1}\right) B_{t-1}}{\Pi_t} + \sum_{j=1}^{J} \frac{L_{j,t-1}}{\Pi_t} \frac{\mathcal{E}_{j,t}}{\mathcal{E}_{j,t-1}} + W_t N_t + \Gamma_t \tag{B.6}$$

The first-order conditions with respect to N_t, B_t, and $L_{j,t}$, are

$$N_t: \quad W_t = \frac{N_t^{\varphi}}{C_t^{-\sigma}} \tag{B.7}$$

$$B_t: \quad C_t^{-\sigma} = \beta \exp\left(i_t\right) \mathrm{E}_t\left[C_{t+1}^{-\sigma} \frac{1}{\Pi_{t+1}} \frac{Z_{t+1}}{Z_t}\right] \tag{B.8}$$

$$L_{j,t}: \quad \frac{L_t^{-\xi}}{C_t^{-\sigma}} = 1 - \beta\, \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1}{\Pi_{t+1}} \frac{\mathcal{E}_{j,t+1}}{\mathcal{E}_{j,t}} \frac{Z_{t+1}}{Z_t}\right] \tag{B.9}$$

Upon log-linearization at the first order around the zero-inflation steady state, the optimality conditions are expressed as:

$$w_t = \varphi\, n_t + \sigma\, c_t \tag{B.10}$$

$$c_t = \mathrm{E}_t[c_{t+1}] - \frac{1}{\sigma}\left(\hat{i}_t - \mathrm{E}_t[\pi_{t+1}]\right) + \frac{1}{\sigma}\left(1-\rho_z\right) z_t \tag{B.11}$$

$$\xi\, l_t - \sigma\, c_t = \frac{\beta}{1-\beta}\left[\sigma\left(c_t - \mathrm{E}_t[c_{t+1}]\right) - \mathrm{E}_t[\pi_{t+1}] + \mathrm{E}_t[\Delta e_{j,t+1}] - \left(1-\rho_z\right) z_t\right] \tag{B.12}$$

B.2 Firms

The production function of firm i that prices in currency j is

$$Y_{j,t}(i) = A_t N_{j,t}(i)^{1-\alpha} \tag{B.13}$$

Recall that W_t denotes the real wage. The real marginal cost of a firm in sector j is then given by

$$\begin{aligned}
\mathrm{MC}_t(i) &= \frac{W_t}{(1-\alpha)\,A_t N_{j,t}(i)^{-\alpha}} && \text{(B.14)}\\
&= \frac{W_t}{(1-\alpha)\,A_t}\left(\frac{Y_{j,t}(i)}{A_t}\right)^{\frac{\alpha}{1-\alpha}} && \text{(B.15)}\\
&= \frac{W_t}{(1-\alpha)\,A_t^{\frac{1}{1-\alpha}}}Y_t^{\frac{\alpha}{1-\alpha}}\left(\frac{\mathcal{E}_{j,t}P_t(i)}{P_t}\right)^{-\frac{\alpha\,\eta}{1-\alpha}} && \text{(B.16)}
\end{aligned}$$

where we have used (11) as well as $C_t = Y_t$ and $C_t(i) = Y_t(i)$ in the last step. The average real marginal cost of the economy

$$\begin{aligned}
\mathrm{MC}_t &= \sum_{j=1}^{J}\int_{V_j(t)} \mathrm{MC}_t(i)di && \text{(B.17)}\\
&= \frac{W_t}{(1-\alpha)\,A_t^{\frac{1}{1-\alpha}}}Y_t^{\frac{\alpha}{1-\alpha}}\left[\sum_{j=1}^{J}\int_{V_{j,t}}\left(\frac{\mathcal{E}_{j,t}P_t(i)}{P_t}\right)^{-\frac{\alpha\,\eta}{1-\alpha}}di\right] && \text{(B.18)}
\end{aligned}$$

In order to log-linearize this equation when $V_{j,t} \equiv V_j$ is independent of time, examine first the term

$$Q_t = \sum_{j=1}^{J}\int_{V_j}\left(\frac{\mathcal{E}_{j,t}P_t(i)}{P_t}\right)^{-\frac{\alpha\,\eta}{1-\alpha}}di$$

Log-linearization delivers

$$-\frac{1-\alpha}{\alpha\eta}\bar{Q}q_t + \left(\sum_{j=1}^{J}\int_{V_j}\left(\frac{\overline{\mathcal{E}_j}\,\overline{P(i)}}{\bar{P}}\right)^{-\frac{\alpha\,\eta}{1-\alpha}}\right)p_t = \sum_{j=1}^{J}\int_{V_j}\left(\frac{\overline{\mathcal{E}_j}\,\overline{P(i)}}{\bar{P}}\right)^{-\frac{\alpha\,\eta}{1-\alpha}}(e_{j,t}+p_t(i))di \tag{B.19}$$

Likewise, log-linearizing the definition of the price index (2) delivers

$$p_t = \sum_{j=1}^{J}\int_{V_j}\left(\frac{\overline{\mathcal{E}_j}\,\overline{P(i)}}{\bar{P}}\right)^{1-\eta}(e_{j,t}+p_t(i))di \tag{B.20}$$

Note now that

$$\frac{\overline{\mathcal{E}_j}\,\overline{P(i)}}{\bar{P}} = 1$$

as all firms will choose the same price expressed in dollars at the steady state, and that price therefore equals the general price index. Use this and (B.20) in (B.19) to find that $q_t = 0$. The log-linearized real marginal cost is therefore

$$\mathrm{mc}_t = w_t + \frac{\alpha}{1-\alpha}\, y_t - \frac{1}{1-\alpha}\, a_t \tag{B.21}$$

It follows that an individual firm's real marginal cost and the average real marginal cost, after log-linearization, are associated with the following equation:

$$\mathrm{mc}_t(i) = \mathrm{mc}_t - \frac{\alpha\,\eta}{1-\alpha}\,[p_t(i) + e_{j,t} - p_t] \tag{B.22}$$

The firm seeks to maximize real profits per its choice of the price $P_t(i)$. With flexible prices, the desired price $P_t(i) = \tilde{P}_{j,t}$ is chosen so as to equalize real marginal revenue to real marginal cost, taking into account the demand function (11) with $C_t = Y_t$ and $C_t(i) = Y_t(i)$. This leads to the markup equation

$$\frac{\eta-1}{\eta}\,\frac{\mathcal{E}_{j,t}\tilde{P}_{j,t}}{P_t} = \mathrm{MC}_t(i) \tag{B.23}$$

where $\mathrm{MC}_t(i)$ is evaluated at the price $P_t(i) = \tilde{P}_{j,t}$. The log-linearized desired price is

$$\tilde{p}_{j,t} = \mathrm{mc}_{j,t|t} + p_t - e_{j,t} \tag{B.24}$$

where (from Eq. (B.22))

$$\mathrm{mc}_{j,t|t} = \mathrm{mc}_t - \frac{\alpha\,\eta}{1-\alpha}\,[\tilde{p}_{j,t} + e_{j,t} - p_t] \tag{B.25}$$

The desired price can be solved as

$$\tilde{p}_{j,t} = \Theta\,\mathrm{mc}_t + p_t - e_{j,t} \tag{B.26}$$

where

$$\Theta = \frac{1-\alpha+\alpha\eta}{1-\alpha} \tag{B.27}$$

Note that the dollar-denominated desired price, $\tilde{p}_{j,t} + e_{j,t}$, is independent of currency choice. When prices are sticky, the log-linearized optimal price is

$$p^*_{j,t} = (1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\mathrm{mc}_{t+\ell|t} + p_{t+\ell} - e_{j,t+\ell}\right] \tag{B.28}$$

To simplify the equation, use Eq. (B.22) to establish the relationship between the optimizing firm's real marginal cost with the economy's average marginal cost:

$$\mathrm{mc}_{t+\ell|t} = \mathrm{mc}_{t+\ell} - \frac{\alpha\,\eta}{1-\alpha}\left[p^*_{j,t} + e_{j,t+\ell} - p_{t+\ell}\right] \tag{B.29}$$

Substitute into Eq. (B.28) and simplify:

$$p^*_{j,t} = (1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\mathrm{mc}_{t+\ell} - \frac{\alpha\,\eta}{1-\alpha}\left[p^*_{j,t} + e_{j,t+\ell} - p_{t+\ell}\right] + p_{t+\ell} - e_{j,t+\ell}\right] \tag{B.30}$$

$$= \Theta^{-1}(1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\Theta\,\mathrm{mc}_{t+\ell} + p_{t+\ell} - e_{j,t+\ell} - (1-\Theta)\,p^*_{j,t}\right] \tag{B.31}$$

$$= -\left(\Theta^{-1}-1\right)p^*_{j,t} + \Theta^{-1}(1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\Theta\,\mathrm{mc}_{t+\ell} + p_{t+\ell} - e_{j,t+\ell}\right] \tag{B.32}$$

$$= (1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\tilde{p}_{j,t+\ell}\right] \tag{B.33}$$

which can be written in a recursive form

$$p^*_{j,t} = \beta\,\theta_j\,\mathrm{E}_t\left[p^*_{j,t+1}\right] + (1-\beta\,\theta_j)\,\tilde{p}_{j,t} \tag{B.34}$$

The law of motion for inflation is derived using the sectoral price index

$$p^*_{j,t} - p_{j,t-1} = \beta\,\theta_j\mathrm{E}_t\left[p^*_{j,t+1} - p_{j,t}\right] + (1-\beta\,\theta_j)\,\tilde{p}_{j,t} - p_{j,t-1} + \beta\,\theta_j\,p_{j,t} \tag{B.35}$$

$$= \beta\,\theta_j\mathrm{E}_t\left[p^*_{j,t+1} - p_{j,t}\right] - (1-\beta\,\theta_j)\,(p_{j,t} - \tilde{p}_{j,t}) + \pi_{j,t} \tag{B.36}$$

From the identity $p_{j,t} = \theta\,p_{j,t-1} + (1-\theta)\,p^*_{j,t}$, we have $\pi_{j,t} = (1-\theta)\left(p^*_{j,t} - p_{j,t-1}\right)$. The above equation becomes

$$(1-\theta)^{-1}\,\pi_{j,t} = (1-\theta)^{-1}\,\beta\,\theta_j\mathrm{E}_t\left[\pi_{j,t+1}\right] - (1-\beta\,\theta_j)\,(p_{j,t} - \tilde{p}_{j,t}) + \pi_{j,t} \tag{B.37}$$

Rearrange terms to obtain the law of motion for sectoral inflation

$$\pi_{j,t} = \beta\,\mathrm{E}_t\left[\pi_{j,t+1}\right] - \lambda_j\,(p_{j,t} - \tilde{p}_{j,t}) \tag{B.38}$$

where $p_{j,t} - \tilde{p}_{j,t}$ is interpreted as the price markup.

B.3 Equilbrium

The market clearing conditions for the goods and labor markets are

$$Y_t = C_t \tag{B.39}$$

where the aggregate output is defined as

$$Y_t \equiv \left(\sum_{j=1}^{J} \int_{V_j} Y_{j,t}(i)^{1-\frac{1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} \tag{B.40}$$

The labor market clears when

$$\begin{aligned} N_t &= \sum_{j=1}^{J} \int_{V_j} N_{j,t}(i) di && \text{(B.41)} \\ &= \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} D_t && \text{(B.42)} \end{aligned}$$

where $D_t \equiv \left[\sum_{j=1}^{J} \int_{\upsilon} \left(\frac{\mathcal{E}_{j,t} P_t(i)}{P_t} \right)^{-\frac{\eta}{1-\alpha}} di \right]$ is a version of price dispersion for a multi-sector economy. The price dispersion is elaborated as

$$\begin{aligned} D_t &= \sum_{j=1}^{J} \left(\frac{\mathcal{E}_{j,t} P_{j,t}}{P_t} \right)^{-\frac{\eta}{1-\alpha}} \int_{V_{j,t}} \left(\frac{P_t(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di && \text{(B.43)} \\ &= \sum_{j=1}^{J} \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} \int_{V_{j,t}} \left(\frac{P_t(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di && \text{(B.44)} \\ &= \sum_{j=1}^{J} \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} D_{j,t} && \text{(B.45)} \end{aligned}$$

where $D_{j,t}$ is the sectoral price dispersion

$$\begin{aligned} D_{j,t} &= \int_{V_{j,t}} \left(\frac{P_t(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di && \text{(B.46)} \\ &= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \int_{V_{j,t} \cap S(i)} \left(\frac{P_{j,t-1}(i)}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} di && \text{(B.47)} \\ &= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \int_{V_{j,t} \cap S(i)} \left(\frac{P_{j,t-1}}{P_{j,t}} \frac{P_{j,t-1}(i)}{P_{j,t-1}} \right)^{-\frac{\eta}{1-\alpha}} di && \text{(B.48)} \\ &= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \Pi_{j,t}^{\frac{\eta}{1-\alpha}} \int_{V_{j,t} \cap S(i)} \left(\frac{P_{j,t-1}(i)}{P_{j,t-1}} \right)^{-\frac{\eta}{1-\alpha}} di && \text{(B.49)} \\ &= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}} \right)^{-\frac{\eta}{1-\alpha}} + \theta_j \Pi_{j,t}^{\frac{\eta}{1-\alpha}} D_{j,t-1} && \text{(B.50)} \end{aligned}$$

The labor market condition is linearized as

$$n_t = \frac{y_t - a_t}{1-\alpha} \tag{B.51}$$

Note that the price dispersion vanishes at the first order.

B.4 Deriving the Sectoral NKPC

Combining (B.10), (B.21), (B.51) and $c_t = y_t$ yields

$$\mathrm{mc}_t = \left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right) y_t - \frac{\phi+1}{1-\alpha} a_t \tag{B.52}$$

With that, (B.26) delivers

$$p_{j,t} - \tilde{p}_{j,t} \quad = \quad \hat{p}_{j,t} - \Theta \left[\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) y_t - \frac{\varphi+1}{1-\alpha}\, a_t\right] \tag{B.53}$$

The last equation eliminates the real wage using the household's optimality condition for the labor supply. Under flexible prices, $p_{j,t} = \tilde{p}_{j,t}$, $\hat{p}_{j,t} = 0$, and we have

$$0 = -\Theta \left[\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) y_t^n - \frac{\varphi+1}{1-\alpha}\, a_t\right] \tag{B.54}$$

Solve Eq. (B.54) for the natural level of output

$$y_t^n = \psi_{ya}\, a_t \tag{B.55}$$

Take the difference between Section B.4 and Eq. (B.54). The price markup is expressed in terms of the output gap defined as $\tilde{y}_t \equiv y_t - y_t^n$:

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) \tilde{y}_t \tag{B.56}$$

Substitute the price markup back into the law of motion for sectoral inflation. The sectoral NKPC is derived as:

$$\pi_{j,t} = \beta\, \mathrm{E}_t\left[\pi_{j,t+1}\right] + \kappa_j\, \tilde{y}_t - \lambda_j\, \hat{p}_{j,t} \tag{B.57}$$

B.5 Dynamic IS Curve

Using the market clearing condition $y_t = c_t$, one can rewrite the optimality condition for government bonds as

$$y_t = \mathrm{E}_t[y_{t+1}] - \frac{1}{\sigma}\left(i_t - \mathrm{E}_t[\pi_{t+1}]\right) + \frac{1}{\sigma}\left(1-\rho_z\right) z_t \tag{B.58}$$

Subtracting the flexible counterpart from B.58 gives

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}\left(i_t - \mathrm{E}_t[\pi_{t+1}] - r_t^n\right) \tag{B.59}$$

where the natural rate of interest is a linear combination of exogenous shocks

$$r_t^n \equiv -\sigma\left(1-\rho_a\right) \psi_{ya} a_t + \left(1-\rho_z\right) z_t \tag{B.60}$$

B.6 NKPC in Non-Linear Form

The firm's profit maximisation problem is expressed as

$$\max_{P^*_{j,t}} \sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[Q_{t,t+\ell} \left[\frac{\mathcal{E}_{j,t+\ell}\, P^*_{j,t}}{P_{t+\ell}} Y_{j,t+\ell|t} - \Psi_{t+\ell}\left(Y_{j,t+\ell|t}\right) \right] \right] \tag{B.61}$$

subject to the demand function

$$Y_{j,t+\ell|t} = \left(\frac{\mathcal{E}_{j,t+\ell}\, P^*_{j,t}}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \tag{B.62}$$

where $\Psi_{t+\ell}(\cdot)$ is the real total cost of production. The first-order condition for price setting is

$$\sum_{\ell=0}^{\infty} {\theta_j}^{\ell} \mathrm{E}_t \left[Q_{t,t+\ell}\, \mathcal{E}_{j,t+\ell}\, (1/P_{t+\ell}) \left(Y_{j,t+\ell|t} + P^*_{j,t} \frac{\partial Y_{j,t+\ell|t}}{\partial P^*_{j,t}} - \frac{P_{t+\ell}\, \mathrm{MC}_{j,t+\ell|t}}{\mathcal{E}_{j,t+\ell}} \frac{\partial Y_{j,t+\ell|t}}{\partial P^*_{j,t}} \right) \right] = 0 \tag{B.63}$$

which can be simplified to the following condition

$$\sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[\beta^{\ell} C_{t+\ell}^{-\sigma} \left(\Pi^*_{j,t} \hat{P}_{j,t} \right)^{-\eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \left(\Pi^*_{j,t} \hat{P}_{j,t+\ell} - \frac{\eta}{\eta-1} \frac{P_{j,t+\ell}}{P_{j,t}} \mathrm{MC}_{j,t+\ell|t} \right) \right] = 0 \tag{B.64}$$

Multiply both sides by $\hat{P}^{\eta}_{j,t}$

$$\sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[\beta^{\ell} C_{t+\ell}^{-\sigma} {\Pi^*_{j,t}}^{-\eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \left(\Pi^*_{j,t} \hat{P}_{j,t+\ell} - \frac{\eta}{\eta-1} \frac{P_{j,t+\ell}}{P_{j,t}} \mathrm{MC}_{j,t+\ell|t} \right) \right] = 0 \tag{B.65}$$

where $\Pi^*_{j,t} \equiv \frac{\mathcal{E}_t P^*_{j,t}}{P_{j,t}}$. Rearrange terms,

$$\sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[\beta^{\ell} C_{t+\ell}^{-\sigma} {\Pi^*_{j,t}}^{1-\eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} \right] \tag{B.66}$$

$$= \sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[\beta^{\ell} C_{t+\ell}^{-\sigma} {\Pi^*_{j,t}}^{-\eta} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} Y_{t+\ell} \frac{\eta}{\eta-1} \mathrm{MC}_{j,t+\ell|t} \right] \tag{B.67}$$

Substitute the expression for idiosyncratic marginal cost:

$$
\Pi_{j,t}^{*\;1+\frac{\alpha\eta}{1-\alpha}} \sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[\beta^{\ell} C_{t+\ell}^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} \frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} Y_{t+\ell} \right] \tag{B.68}
$$

$$
= \frac{\eta}{\eta-1} \sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[\beta^{\ell} C_{t+\ell}^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\eta} \left(\frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} \right)^{-\frac{\alpha\eta}{1-\alpha}} Y_{t+\ell} \, \mathrm{MC}_{t+\ell} \right] \tag{B.69}
$$

which can be rewritten as

$$
\Pi_{j,t}^{*\;1+\frac{\alpha\eta}{1-\alpha}} x_{2,t} = \frac{\eta}{\eta-1} x_{1,t} \tag{B.70}
$$

where

$$
x_{2,t} = C_t^{-\sigma} Y_t \hat{P}_{j,t} + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\eta-1} \left(\frac{\hat{P}_{j,t+1}}{\hat{P}_{j,t}} \right)^{-\eta} x_{2,t+1} \right] \tag{B.71}
$$

and

$$
x_{1,t} = C_t^{-\sigma} Y_t \hat{P}_{j,t}^{-\frac{\alpha\eta}{1-\alpha}} \mathrm{MC}_t + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\frac{\eta}{1-\alpha}} \left(\frac{\hat{P}_{j,t+1}}{\hat{P}_{j,t}} \right)^{-\eta} x_{1,t+1} \right] \tag{B.72}
$$

The relative price follows

$$
\hat{P}_{j,t} = \hat{P}_{j,t-1} \frac{\Pi_t^{e} \Pi_{j,t}}{\Pi_t} \tag{B.73}
$$

In the case of flexible prices, the desired price is given by

$$
\tilde{P}_{j,t} = \frac{P_t}{\mathcal{E}_{j,t}} \left(\frac{\eta}{\eta-1} \mathrm{MC}_t \right)^{\Theta} \tag{B.74}
$$

B.7 Value Function in Non-Linear Form

Iterate the value function:

$$
\mathcal{V}_{j,t|t} = \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \Xi_{j,t+\ell|t} \right] \tag{B.75}
$$

$$
= \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^{*}}{P_{t+\ell}} Y_{j,t+\ell|t} \right] - \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \mathrm{MC}_{j,t+\ell|t} Y_{j,t+\ell|t} \right]
$$

The value function of the firm is therefore

$$
\mathcal{V}_{j,t} = \mathcal{R}_{j,t} - \mathcal{C}_{j,t} \tag{B.76}
$$

The revenue is given by

$$\mathcal{R}_{j,t} = \sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \tag{B.77}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \tag{B.78}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}}\right)^{-\eta} Y_{t+\ell} \tag{B.79}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}}\right)^{1-\eta} Y_{t+\ell} \tag{B.80}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\eta} Y_{t+\ell} \tag{B.81}$$

$$= \frac{1}{C_t^{-\sigma}} {\Pi_{j,t}^*}^{1-\eta} \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\eta} Y_{t+\ell} \tag{B.82}$$

Let $\tilde{\mathcal{R}}_{j,t} \equiv \mathcal{R}_{j,t} {\Pi_{j,t}^*}^{\eta-1} C_t^{-\sigma}$, then

$$\tilde{\mathcal{R}}_{j,t} = \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} \mathrm{E}_t \left[C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\eta} Y_{t+\ell} \right] \tag{B.83}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \sum_{\ell=1}^{\infty} (\beta\theta_j)^{\ell} \mathrm{E}_t \left[C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\eta} Y_{t+\ell} \right] \tag{B.84}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \mathrm{E}_t \left[\left(\frac{P_{j,t}}{P_{j,t+1}}\right)^{1-\eta} \sum_{\ell=1}^{\infty} (\beta\theta_j)^{\ell} C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t+1}}{P_{j,t+\ell}}\right)^{1-\eta} Y_{t+\ell} \right] \tag{B.85}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \beta\theta_j \mathrm{E}_t \left[\left(\frac{P_{j,t}}{P_{j,t+1}}\right)^{1-\eta} \tilde{\mathcal{R}}_{j,t+1} \right] \tag{B.86}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\eta} Y_t + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\eta-1} \tilde{\mathcal{R}}_{j,t+1} \right] \tag{B.87}$$

The discounted cost is

$$\mathcal{C}_{j,t} = \sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} Y_{j,t+\ell|t} \mathrm{MC}_{j,t+\ell|t} \tag{B.88}$$

$$= \sum_{\ell=0}^{\infty} \theta_j^{\ell} \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\Pi_{j,t}^{*} \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\eta} Y_{t+\ell} \left(\Pi_{j,t}^{*} \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\alpha\eta}{1-\alpha}} \mathrm{MC}_{j,t+\ell} \tag{B.89}$$

$$= \sum_{\ell=0}^{\infty} \theta_j^{\ell} \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\Pi_{j,t}^{*} \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\eta}{1-\alpha}} Y_{t+\ell} \mathrm{MC}_{j,t+\ell} \tag{B.90}$$

$$= \frac{1}{C_t^{-\sigma}} \Pi_{j,t}^{*}{}^{-\frac{\eta}{1-\alpha}} \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\eta}{1-\alpha}} Y_{t+\ell} \mathrm{MC}_{t+\ell} \tag{B.91}$$

Let $\tilde{\mathcal{C}}_{j,t} \equiv \mathcal{C}_{j,t} C_t^{-\sigma} \Pi_{j,t}^{*}{}^{\frac{\eta}{1-\alpha}}$, then

$$\tilde{\mathcal{C}}_{j,t} = \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\eta}{1-\alpha}} Y_{t+\ell} \mathrm{MC}_{t+\ell} \tag{B.92}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} Y_t \mathrm{MC}_t + \beta\theta_j \mathrm{E}_t \left(\frac{P_{j,t}}{P_{j,t+1}}\right)^{-\frac{\eta}{1-\alpha}} \tilde{\mathcal{C}}_{j,t+1} \tag{B.93}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{-\frac{\eta}{1-\alpha}} Y_t \mathrm{MC}_t + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\frac{\eta}{1-\alpha}} \tilde{\mathcal{C}}_{j,t+1}\right] \tag{B.94}$$

The steady states are

$$\mathcal{V}_j = \mathcal{V} = \frac{Y(1-\mathrm{MC})}{1-\beta\theta} \tag{B.95}$$

Appendix C Solutions to a Two-Sector Economy

C.1 AIAO

Since the exchange rate shock is the only exogenous variable in the equation system, all endogenous variables can be expressed in terms of it:

$$\tilde{y}_t = \psi_{ye}^{aiao} \Delta e_t; \quad \pi_t = \psi_{\pi e}^{aiao} \Delta e_t; \quad \hat{i}_t = \psi_{ie}^{aiao} \Delta e_t. \tag{C.1}$$

where ψ_{ye}^{aiao}, $\psi_{\pi e}^{aiao}$, and ψ_{ie}^{aiao} are unknown coefficients to be determined. Because $\mathrm{E}_t \Delta e_{t+1} = 0$, the forecasts of the endogenous variables one period ahead are:

$$\mathrm{E}_t[\tilde{y}_{t+1}] = 0; \quad \mathrm{E}_t[\pi_{t+1}] = 0. \tag{C.2}$$

Substitute $\mathrm{E}_t[\tilde{y}_{t+1}]$ and $\mathrm{E}_t[\pi_{t+1}]$ into the three-equation system, and use the Taylor rule to substitute out the nominal interest rate in the dynamic IS curve. The equation system reduces to:

$$\tilde{y}_t = -\sigma^{-1} \left(\phi_\pi \pi_t + \phi_y \tilde{y}_t \right) \tag{C.3}$$
$$\pi_t = \kappa \tilde{y}_t + \upsilon \Delta e_t \tag{C.4}$$

from which π_t and $\tilde{y}_t$ can be solved in terms of Δe_t:

$$\tilde{y}_t = -\frac{\upsilon \phi_\pi}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t \tag{C.5}$$
$$\pi_t = \frac{\upsilon \left(\sigma + \phi_y \right)}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t \tag{C.6}$$

From the Taylor rule, the nominal interest rate is

$$\hat{i}_t = \frac{\upsilon \sigma \phi_\pi}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t. \tag{C.7}$$

Define $\Omega \equiv \sigma + \phi_y + \kappa \phi_\pi$. The coefficients are

$$\psi_{ye}^{aiao} = -\upsilon \phi_\pi \Omega, \tag{C.8}$$
$$\psi_{\pi e}^{aiao} = \upsilon \left(\sigma + \phi_y \right) \Omega, \tag{C.9}$$
$$\psi_{ie}^{aiao} = \upsilon \sigma \phi_\pi \Omega. \tag{C.10}$$

The sectoral dynamics involve the bilateral relative price as a state variable.

$$\pi_{1,t} = \psi_{\pi_1 s}^{aiao} s_{t-1} + \psi_{\pi_1 e}^{aiao} \Delta e_t; \quad \pi_{2,t} = \psi_{\pi_2 s}^{aiao} s_{t-1} + \psi_{\pi_2 e}^{aiao} \Delta e_t. \tag{C.11}$$

The forecasts for the sectoral inflation are

$$\mathrm{E}_t \pi_{1,t+1} = \psi_{\pi_1 s}^{aiao} s_t; \quad \mathrm{E}_t \pi_{2,t+1} = \psi_{\pi_2 s}^{aiao} s_t \tag{C.12}$$

From the dollar sector's NKPC:

$$\pi_{1,t} = \beta\psi^{aiao}_{\pi_1 s} s_t + \kappa\psi^{aiao}_{ye}\Delta e_t + \lambda\upsilon s_t \tag{C.13}$$
$$= \left(\beta\psi^{aiao}_{\pi_1 s} + \lambda\upsilon\right) s_t + \kappa\psi^{aiao}_{ye}\Delta e_t \tag{C.14}$$
$$= \left(\beta\psi^{aiao}_{\pi_1 s} + \lambda\upsilon\right)\theta s_{t-1} + \left[\left(\beta\psi^{aiao}_{\pi_1 s} + \lambda\upsilon\right)\theta + \kappa\psi^{aiao}_{ye}\right]\Delta e_t \tag{C.15}$$

The last equation comes from the autoregressive representation of the bilateral relative price. From comparison with the unknown coefficients:

$$\psi^{aiao}_{\pi_1 s} = \left(\beta\psi^{aiao}_{\pi_1 s} + \lambda\upsilon\right)\theta \tag{C.16}$$
$$\psi^{aiao}_{\pi_1 e} = \left(\beta\psi^{aiao}_{\pi_1 s} + \lambda\upsilon\right)\theta + \kappa\psi^{aiao}_{ye} \tag{C.17}$$

The coefficients $\psi^{aiao}_{\pi_1 s}$ and $\psi^{aiao}_{\pi_1 e}$ can be solved as:

$$\psi^{aiao}_{\pi_1 s} = \upsilon\left(1-\theta\right) \tag{C.18}$$
$$\psi^{aiao}_{\pi_1 e} = \upsilon\left(1-\theta\right) - \upsilon\kappa\phi_\pi\Omega \tag{C.19}$$

By combining the terms with common coefficients, the dollar-sector inflation can be rewritten as a function of the contemporary bilateral price and the exchange rate shock:

$$\pi_{1,t} = \frac{\upsilon\left(1-\theta\right)}{\theta} s_t - \upsilon\kappa\phi_\pi\Omega\Delta e_t \tag{C.20}$$
$$= \upsilon\left(1-\theta\right) s_{t-1} + \upsilon\left(1-\theta-\kappa\phi_\pi\Omega\right)\Delta e_t \tag{C.21}$$

The dollar-sector inflation is above its steady state if

$$1-\theta > \kappa\phi_\pi\Omega \tag{C.22}$$
$$\sigma + \phi_y + \kappa\phi_\pi > \frac{\kappa}{1-\theta}\phi_\pi \tag{C.23}$$
$$\sigma + \phi_y > \phi_\pi\left(1-\beta\theta\right)\Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) \tag{C.24}$$
$$\left(1-\beta\theta\right)\Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) < \frac{\sigma+\phi_y}{\phi_\pi} \tag{C.25}$$

From Proposition 2, inflation in the non-dollar sector is

$$\pi_{2,t} = \pi_{1,t} - \frac{1-\theta}{\theta} s_t \tag{C.26}$$
$$= -\frac{\left(1-\upsilon\right)\left(1-\theta\right)}{\theta} s_t - \upsilon\kappa\phi_\pi\Omega\Delta e_t \tag{C.27}$$

The sectoral output gaps are derived from the demand functions:

$$
\begin{aligned}
\tilde{y}_{1,t} &= \tilde{y}_t + \upsilon\eta s_t && \text{(C.28)}\\
&= \upsilon\eta s_t - \upsilon\phi_\pi\Omega\Delta e_t && \text{(C.29)}\\
&= \upsilon\eta\theta s_{t-1} + \upsilon\left(\eta\theta - \phi_\pi\Omega\right)\Delta e_t && \text{(C.30)}\\
\tilde{y}_{2,t} &= \tilde{y}_t - \left(1-\upsilon\right)\eta s_t && \text{(C.31)}\\
&= -\left(1-\upsilon\right)\eta s_t - \upsilon\phi_\pi\Omega\Delta e_t && \text{(C.32)}
\end{aligned}
$$

The contemporaneous response of the dollar-sector output gap depends on the parameters. The sector produces above the natural level if

$$
\begin{aligned}
\eta\theta &> \phi_\pi\Omega && \text{(C.33)}\\
\sigma + \phi_y + \kappa\phi_\pi &> \frac{\phi_\pi}{\eta\theta} && \text{(C.34)}\\
\left(\frac{1}{\eta\theta} - \kappa\right)\phi_\pi &< \sigma + \phi_y && \text{(C.35)}\\
\frac{1}{\eta\theta} - \kappa &< \frac{\sigma + \phi_y}{\phi_\pi} && \text{(C.36)}
\end{aligned}
$$

C.2 DIAO

The expected aggregate inflation in the dynamic IS curve is rewritten as:

$$
\begin{aligned}
\mathrm{E}_t[\pi_{t+1}] &= \mathrm{E}_t\left[\left(1-\upsilon\right)\pi_{1,t+1} + \upsilon\pi_{2,t+1}\right] && \text{(C.37)}\\
&= \mathrm{E}_t\left[\pi_{1,t+1} + \upsilon\left(s_{t+1} - s_t\right)\right] && \text{(C.38)}\\
&= \mathrm{E}_t\left[\pi_{1,t+1}\right] - \upsilon\left(1-\theta\right)s_t && \text{(C.39)}
\end{aligned}
$$

where the last equation follows from Eq. (45). The bilateral relative price s_t can be viewed as an exogenous autoregressive variable in a three-equation system; so all endogenous variables can be written as functions of s_t. Let $\tilde{y}_t = \psi_{ys}^{diao} s_t$, and $\pi_{1,t} = \psi_{\pi_1 s}^{diao} s_t$. Again, it follows from Eq. (45) that $\mathrm{E}_t[\tilde{y}_{t+1}] = \psi_{ys}^{diao}\theta s_t$, and $\mathrm{E}_t[\pi_{1,t+1}] = \psi_{\pi_1 s}^{diao}\theta s_t$. Substitute these, together with the Taylor rule, into the dynamic IS curve and the dollar-sector NKPC yields the following equation system for ψ_{ys}^{diao} and $\psi_{\pi_1 s}^{diao}$:

$$
\begin{aligned}
\left(1-\beta\theta\right)\psi_{\pi_1 s}^{diao} &= \kappa\psi_{ys}^{diao} + \lambda\upsilon && \text{(C.40)}\\
\left(\phi_\pi - \theta\right)\psi_{\pi_1 s}^{diao} &= -\left[\sigma\left(1-\theta\right) + \phi_y\right]\psi_{ys}^{diao} - \upsilon\left(1-\theta\right) && \text{(C.41)}
\end{aligned}
$$

Define $\Lambda \equiv \frac{1}{(1-\beta\theta)[\sigma(1-\theta)+\phi_y]+\kappa(\phi_\pi-\theta)}$. The solutions to the equation system are:

$$
\begin{aligned}
\psi_{ys}^{diao} &= -\lambda\upsilon\phi_\pi\Lambda && \text{(C.42)}\\
\psi_{\pi_1 s}^{diao} &= \frac{\upsilon\left(1-\theta\right)}{\theta}\left(1 - \kappa\phi_\pi\Lambda\right) && \text{(C.43)}
\end{aligned}
$$

The response of dollar-sector inflation depends on the parameters. It is negative if $\psi_{\pi_1 s}^{diao} < 0$, and positive otherwise. Taking into account the expression of Λ, this condition becomes:

$$(1-\beta\theta)\left[\sigma(1-\theta)+\phi_y\right]+\kappa(\phi_\pi-\theta)<\kappa\phi_\pi \tag{C.44}$$
$$(1-\beta\theta)\left[\sigma(1-\theta)+\phi_y\right]<\kappa\theta \tag{C.45}$$
$$\lambda\sigma+\frac{(1-\beta\theta)\,\phi_y}{\theta}<\kappa \tag{C.46}$$
$$\phi_y<\frac{(\kappa-\lambda\sigma)\,\theta}{1-\beta\theta} \tag{C.47}$$

The dynamics of the nominal interest rate are obtained by substituting the solutions to the output gap and dollar-sector inflation into the Taylor rule:

$$\hat{i}_t=\phi_\pi\psi^{diao}_{\pi_1 s}s_t+\phi_y\psi^{diao}_{ys}s_t \tag{C.48}$$
$$=-\upsilon(\kappa-\lambda\sigma)(1-\theta)\,\phi_\pi\Lambda s_t \tag{C.49}$$

Its response is negative if:

$$\kappa\sigma^{-1}>\lambda \tag{C.50}$$
$$\lambda\Theta\left(1+\frac{\sigma^{-1}(\alpha+\varphi)}{1-\alpha}\right)>\lambda \tag{C.51}$$
$$\sigma^{-1}>\frac{1-\Theta}{\Theta}\frac{1-\alpha}{\alpha+\varphi} \tag{C.52}$$

Inflation in the non-dollar sector is obtained using Proposition 2:

$$\pi_{2,t}=\pi_{1,t}-\frac{1-\theta}{\theta}s_t \tag{C.53}$$
$$=-\frac{1-\theta}{\theta}\left(1-\upsilon+\upsilon\kappa\phi_\pi\Lambda\right)s_t \tag{C.54}$$

The aggregate inflation is the weighted sum of the sectoral inflation:

$$\pi_t=(1-\upsilon)\,\pi_{1,t}+\upsilon\pi_{2,t} \tag{C.55}$$
$$=-\frac{1-\theta}{\theta}\upsilon\kappa\phi_\pi\Lambda s_t+\upsilon\Delta e_t \tag{C.56}$$
$$=-(1-\theta)\,\upsilon\kappa\phi_\pi\Lambda s_{t-1}+\upsilon\left[1-(1-\theta)\,\kappa\phi_\pi\Lambda\right]\Delta e_t \tag{C.57}$$
$$\tag{C.58}$$

The coefficient of the exchange rate shock is

$$\upsilon\left\{\kappa\theta(\phi_\pi-1)+(1-\beta\theta)\left[\sigma(1-\theta)+\phi_y\right]\right\}\Lambda>0 \tag{C.59}$$

The real interest rate is:

$$\hat{r}_t = \hat{i}_t - \mathrm{E}_t[\pi_{t+1}] \tag{C.60}$$

$$= -\upsilon\left(\kappa - \lambda\sigma\right)\left(1-\theta\right)\phi_\pi \Lambda s_t + \frac{1-\theta}{\theta}\upsilon\kappa\phi_\pi\Lambda\theta s_t \tag{C.61}$$

$$= \upsilon\lambda\sigma\left(1-\theta\right)\phi_\pi\Lambda s_t \tag{C.62}$$

The sectoral output gap dynamics can be derived from the demand functions:

$$\tilde{y}_{1,t} = \tilde{y}_t + \upsilon\eta s_t = -\upsilon\left(\lambda\phi_\pi\Lambda - \eta\right)s_t \tag{C.63}$$

$$\tilde{y}_{2,t} = \tilde{y}_t - \left(1-\upsilon\right)\eta s_t = -\left[\lambda\upsilon\phi_\pi\Lambda + \eta\left(1-\upsilon\right)\right]s_t \tag{C.64}$$

For the dollar sector to produce above the natural level, $\eta > \lambda\phi_\pi\Lambda$. Otherwise, it produces below the natural level.

C.3 DIDO

Follow the same method as in the case of DIAO. Solutions to the three-equation system are:

$$\tilde{y}_t = -\lambda\,\upsilon\,\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\,\phi_y\right)\Lambda\,s_t \tag{C.65}$$

$$\pi_{1,t} = \frac{\upsilon\,\left(1-\theta\right)}{\theta}\left[1-\kappa\,\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\phi_y\right)\Lambda\right]\,s_t \tag{C.66}$$

$$\hat{i}_t = -\upsilon\,\left(\kappa - \lambda\,\sigma\right)\left(1-\theta\right)\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\,\phi_y\right)\Lambda\,s_t \tag{C.67}$$

From the IS curve, the real interest rate is positive:

$$\hat{r}_t = \upsilon\,\lambda\,\sigma\,\left(1-\theta\right)\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\,\phi_y\right)\Lambda\,s_t \tag{C.68}$$

Inflation in the non-dollar sector is negative:

$$\pi_{2,t} = -\frac{1-\theta}{\theta}\left[1-\upsilon+\upsilon\,\kappa\,\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\phi_y\right)\Lambda\right]\,s_t < 0 \tag{C.69}$$

Sectoral output gaps are given by:

$$\tilde{y}_{1,t} = -\upsilon\,\left[\lambda\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\,\phi_y\right)\Lambda - \eta\right]\,s_t \tag{C.70}$$

$$\tilde{y}_{2,t} = -\left[\lambda\,\upsilon\,\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\,\phi_y\right)\,\Lambda + \eta\left(1-\upsilon\right)\right]\,s_t \tag{C.71}$$

Aggregate inflation is:

$$\pi_t = -\frac{1-\theta}{\theta}\,\upsilon\,\kappa\,\left(\phi_\pi + \frac{\theta\,\eta}{1-\theta}\,\phi_y\right)\,\Lambda\,s_t + \upsilon\,\Delta e_t \tag{C.72}$$

The coefficient of the exchange rate shock is

$$\upsilon \left\{ \kappa\theta \left(\phi_\pi - 1 - \eta\phi_y \right) + \left(1 - \beta\theta \right) \left[\sigma \left(1 - \theta \right) + \phi_y \right] \right\} \Lambda \tag{C.73}$$

whose sign depends on the parameters. In the case of a large elasticity of substitution, the aggregate inflation response is negative.