

The Intangible Gap

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Abstract

We document a large and rising intangible investment gap between small and large U.S. firms. Smaller firms invest disproportionately in intangible capital, despite facing tighter financing constraints, and this gap has tripled since the 1980s alongside a pronounced increase in intangible investment volatility. We develop a dynamic industry-equilibrium model in which firms invest in both physical and intangible capital under financial frictions. Intangible investment is subject to idiosyncratic quality shocks and can be partially financed internally through equity-based compensation. These features generate an option-like payoff to intangible investment: downside risk is limited by exit, while upside gains scale with realized quality. This mechanism makes intangible capital particularly attractive for small firms with high exit risk. Our analysis reveals that the joint increase in intangible investment volatility and the decline in financing frictions for intangible capital account for approximately 60% of the post-2000 gap in intangible-to-physical capital ratios between small and large firms over the 2001–2023 period.

Keywords: Investment, Intangible, Financial Friction, Size Distribution

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1 Introduction

One of the most prominent ongoing transformations in the US corporate sector is the change in its nature of corporate investment. The introduction of new technology has led to a structural change towards a more intangible-intensive economy (Corrado and Hulten (2010)). Intangible investment rose remarkably (Falato, Kadyrzhanova, Sim, and Steri (2022), Crouzet and Eberly (2019), Corrado, Hulten, and Sichel (2009)) since the 1980s, despite the weak physical capital investment (Alexander and Eberly (2018), Gutiérrez and Philippon (2017), Hall (2015)). At the same time, a key feature of intangible investment is its high and rising uncertainty. Intangible capital—such as R&D, human capital, and organizational capital—typically involves substantial uncertainty in both outcomes and valuation. The investment process itself has also become increasingly costly and volatile. Recent examples highlight this trend: innovation pipelines have shifted toward highly stochastic processes. These innovations generate extremely skewed outcomes—ranging from breakthrough performance to complete training collapse—implying that the variance of innovation shocks has increased dramatically.

In this paper, we document two new stylized facts about firms' intangible investment. First, there is a substantial *intangible gap* between small and large firms: small firms tend to have a higher ratio of intangible over physical capital even though smaller firms tend to be financially constrained. Smaller firms have higher growth rates, higher growth volatility, and their intangible investments are more volatile than physical investments. However, we do not observe a distinct pattern in the physical capital investment between firms over the sample period. Smaller firms contribute to both the level and the volatility of intangible investment substantially. This intangible gap has grown dramatically over time. Second, we document a pronounced increase in both the volatility of intangible investment and the uncertainty in the valuation of intangible capital within the universe of publicly traded firms.

These stylized facts are not direct implications of standard firm-dynamics models—such as Cooley and Quadrini (2001) with two capital inputs—and it is not clear that the rise in intangible intensity alone provides a natural explanation. While a change in the aggregate production function can account for the increasing share of intangible capital in the economy,

it does not explain the pronounced heterogeneity in firms' behavior.

We propose an equilibrium model with firm dynamics (Hopenhayn (1992), Cooley and Quadrini (2001), and Gomes (2001)) to study firms' investment choices on intangible and physical capital over their life-cycles. In this economy, firms make investment decisions in both intangible and physical capital with financial constraints. As standard in the model with firm dynamics, productive firms invest in both intangible and physical capital and grow larger, while small and unproductive firms can exit. Our crucial assumption is that intangible and physical investments face *heterogeneous* investment opportunity sets. Intangible capital accumulation is subject to a random shock, ω , which affects the quality of the intangible output and hence the effective input in the production function. In contrast, physical capital accumulation follows a standard law of motion. We call this shock the intangible *quality* shock. The risk in the intangible investment has an impact on firms' investment decisions. The quality shock is a stochastic adjustment cost, making intangible investment risky. Firms on average deviate from their benchmark intangible investment level.¹ However, the randomness in the intangible capital's productivity introduced additional real option value, the value of the option to exit. Since intangible capital is risky, the option for small firms to exit becomes more valuable. For a given distribution of intangible quality shock, smaller firms invest more in intangible capital relative to physical capital than larger firms.

What drives the difference in the investment opportunity sets for intangible and physical capital? Intangible investments are different from physical investments in nature: intangible investments are considerably riskier than physical investment, and they are unpledgeable so harder to get external financing. In the old economy, firms primarily invested in physical for expansion, e.g., things like buildings and machines which are tangible and easy to estimate their value. However, it is more difficult to foresee the value of intangible capital. A new patent resulting from a sequence of R&D investments may yield highly uncertain returns due to unpredictable demand. For instance, Large language model (LLM) development has moved away from predictable, incremental transformer training and toward reinforcement-learning

¹The benchmark level is the investment rate without the intangible quality shock. The average intangible investment rate can be higher than the benchmark level if the mean of the quality shock is higher than one.

pipelines—such as DeepSeek’s RLOO—that exhibit significantly greater variance and instability. In addition to patent innovation, other crucial components of intangible capital, e.g., organizational structure, management, and key talents’ skill and mobility, are subject to great uncertainty. The risks of organizational structure becoming obsolete and the leaving of key talents impose challenges in the development of firms’ intangible capital. Given the differences, small and large firms may even differ in their motivations for investing in intangibles.

We examine the publicly listed firms in the US for our analysis because of the availability of detailed financial data of the public firms. Different from patent innovations, components of intangible capital might not always be directly measurable, recorded separately in the financial statements², they are reported as part of the selling, general and administrative expenses in the cash flow statement, or R&D expenses if the activities are research and development related. Following the literature, we then estimate the accumulated intangible capital using the perpetual inventory method. To capture the dynamics of intangible capital, we construct the relative capital ratio, *hkratio*, for each firm year.

Over the sample period from 1975 to 2023, we observe substantial growth in intangible capital over time. The average *hkratio* grew more than three times since the 1980s. In the meantime, we see much more rapid growth in intangible capital in the smallest firm size group. Since the 1980s, the accumulated investment in intangible capital has increased by more than 7.1 times relative to physical investment on average among the smallest firms. In contrast, the increase is only 2.2 times among the largest firms. However, we do not find a similar pattern using the balance sheet intangible assets, which indicates that larger firms accumulate intangible assets mostly through acquisition. We also find that the volatility of *hkratio* and the volatility of patent market values rose dramatically over the sample period, consistent with increasing randomness in intangible asset quality. The firm-level volatility of *hkratio* has surged among small firms, rising from 0.15 in 1980 to 2.55 in 2023, while remaining largely stable for large firms. A similar pattern emerges in the market-based measure: the cross-sectional dispersion

²For example, in addition to management skills, organizational structure, founder’s influences when firms are young, workers’ stock and option incentives are not always reflected in firms’ current financial statements. (Eisfeldt, Falato, and Xiaolan, 2023) proposed a balance sheet-based measure for employees’ stock-based compensation for a long sample period.

in patent values has increased sharply since the early 1980s. These findings provide direct empirical support for modeling intangible capital accumulation as subject to idiosyncratic quality shocks, a central feature of our theoretical framework.

Our model highlights two central mechanisms that jointly generate the intangible gap in the data. The first is the real option effects created by the intangible quality shock, ω . This assumption is grounded in our finding that the rising cross-sectional variance of intangible intensity—not simply an upward shift in the typical firm—is a primary driver of the aggregate increase in intangible capital ratios. Because intangible capital is subject to random scaling through ω , firms face a convex payoff structure: high-quality realizations of ω substantially raise effective intangible capital, while low realizations trigger exit and limit losses. This asymmetry makes intangible investment particularly attractive for small firms, which place high value on the option to exit. Consequently, small firms optimally tilt their portfolios toward intangible capital relative to large firms. The second mechanism operates through the relatively relaxed financing frictions for intangibles compared with physical capital. Intangible capital can be financed internally through the labor market—for example, via equity-based compensation—which effectively allows firms to borrow from their workers (Sun and Xiaolan, 2019; Guiso, Pistaferri, and Schivardi, 2012). A higher stock of intangible capital relaxes financing constraints and encourages further intangible accumulation. Together, the real-option mechanism and the financing channel reinforce each other: rising uncertainty increases the option value of intangible investment, while improved financing capacity allows firms—especially smaller ones—to take advantage of this option. This interaction generates a skewed distribution of intangible intensity across firms and amplifies dispersion in investment behavior.

Our theoretical mechanism makes a twofold contribution. First, we extend the classic firm-dynamics framework with two capital inputs (e.g., Cooley and Quadrini (2001); Gomes (2001)) by introducing a mechanism that generates the cross-sectional heterogeneity in intangible intensity observed in the data: a multiplicative intangible-quality shock ω , combined with the firm's option to exit when ω is low, generates a convex payoff to intangible investment specifically, while physical investment remains subject to the standard real-options logic. Second, our mechanism contributes to the literature on the uncertainty-investment relationship. Existing

work has focused on physical capital and largely documents a negative relationship between uncertainty and investment. Our mechanism instead generates a positive uncertainty-investment relationship for intangible capital while leaving physical investment subject to the standard negative-sign result, thereby producing a *differential* gap between intangible and physical investment.

We calibrate the model to match key moments from the pre-2000 U.S. data—specifically, the cross-sectional spread in *hkratio* and the share of employee equity in financing intangible capital. The benchmark parameters successfully replicate several salient features: the pre-2000 H/K spread (the difference in *hkratio* between small and large firms), the share of employee equity in firm value, relative investment rates in H and K, and the average firm exit rate. Importantly, the model also reproduces the monotonic decline in *hkratio* across size groups.

We then conduct comparative statics to assess how rising volatility of intangible quality contributes to the post-2000 rise in intangible intensity. Increasing the volatility of the quality shock by 37.5%—a magnitude consistent with the documented rise in patent-value dispersion—substantially amplifies the real-option value of intangible investment. Small firms raise their intangible-to-physical capital ratios more than large firms. Quantitatively, the model-implied H/K spread increases from 0.503 in the benchmark to 0.812 under the higher volatility calibration. This accounts for approximately 60% of the observed post-2000 H/K spread of 1.374.

Our findings are robust to allowing physical capital to relax financing constraints alongside intangible capital. Using [Lian and Ma \(2021\)](#) on asset-based corporate borrowing, the plausible range of physical-capital pledgeability is $\theta_k \in [0.05, 0.10]$, an order of magnitude below the intangible-capital pledgeability $\theta_h = 0.8$ in our baseline. Throughout this empirically reasonable range, the model continues to generate a substantial size-dependent intangible gap, with the H/K amplification weakening only modestly. The asymmetric treatment of intangible and physical capital in our model is therefore not a modeling convenience but reflects a quantitatively meaningful difference in how firms finance the two types of investment.

Our findings highlight a broader transformation in the U.S. corporate sector: intangible investment is increasingly concentrated among smaller, younger, and lower-valuation firms.

These firms bear a substantial share of the economy's innovation risk, undertake more intangible investment, and disproportionately drive the cross-sectional dispersion in investment behavior.

Literature Review Our paper is related to the growing literature on the rise of intangible capital in the US economy. [Crouzet and Eberly \(2019\)](#), [Alexander and Eberly \(2018\)](#), [Ayyagari, Demirgüç-Kunt, and Maksimovic \(2024\)](#) provide links between the increasing importance of intangible capital and the rise of concentration. [Falato et al. \(2022\)](#) studies the relationship between companies' cash holdings and the rise of intangibles. [Crouzet and Eberly \(2023\)](#) evaluates the implications of firms shifting investment to intangibles on market power, valuation, and investment incentives. Our paper focuses on the heterogeneous growth patterns in intangible investment across firm size. We show that frictions in intangible investment must be modeled differently in order to capture the observed patterns of intangible investment under financial constraints.

Our project is related to a vast literature of understanding firm dynamics, size distribution, age, and growth ([Lucas and Prescott, 1971](#); [Lucas, 1978](#); [Hopenhayn, 1992](#); [Hopenhayn and Rogerson, 1993](#)). In these models, they characterize the stationary firm size distribution and investment policies. [Cooley and Quadrini \(2001\)](#) study the relationships among firm size, age, and financial borrowing constraints. Small and young firms grow faster, create and destroy more jobs, and are more financially constrained. [Clementi and Palazzo \(2016\)](#) find that the entry and exit margins account for 20% of output growth over ten years. The extensive margin also generates persistence in aggregate time series. We introduce an intangible quality shock in firms' investment decisions and expand the model to better understand the joint distribution of firm size, intangible, and physical capital. [Ottonello and Winberry \(2024\)](#) find that over the firm life cycle, firms increasingly invest in R&D, whereas younger and smaller firms initially concentrate on physical investment. Our findings, in contrast, focus on the level and evolution of the cumulative intangible capital of small versus large firms and on the relative changes in the investment gap over time. We highlight the role of financial frictions, consistent with the existing literature on firms' capital allocation choices, but introduce a novel mechanism:

intangible capital is subject to additional quality uncertainty. This feature helps explain both the magnitude and the persistent rise of the intangible gap observed in the data.

Our paper also contributes to the broader literature on the uncertainty–investment relationship. The canonical real-options framework—[Bernanke \(1983\)](#), [Pindyck \(1988, 1991\)](#), [Caballero \(1991\)](#), [Caballero and Pindyck \(1996\)](#), and [Dixit and Pindyck \(1994\)](#)—predicts that under irreversibility, higher uncertainty raises the option value of waiting and depresses investment. This literature has primarily focused on *physical* capital, and the empirical work supports the negative relationship in that context: [Leahy and Whited \(1996\)](#) document it at the firm level and [Bloom \(2009\)](#) extends it to aggregate uncertainty shocks. On the other hand, a literature also identifies settings in which the sign can flip: [Eisdorfer \(2008\)](#) shows that financially distressed firms invest more under high uncertainty because debt-financed equity has a convex payoff in distress (a risk-shifting channel), and [Ferreira, Manso, and Silva \(2014\)](#) argue that private ownership encourages innovation by giving insiders greater tolerance for early failure. These channels, however, affect all forms of investment uniformly within an affected firm, and therefore cannot generate a *differential* gap between physical and intangible investment. Explaining the size-dependent intangible gap documented in [Section 2.3](#) requires a mechanism whose effect on intangible investment differs from its effect on physical investment within the same firm. We propose such a mechanism: a multiplicative intangible-quality shock ω , combined with the firm’s option to exit when ω is low, generates a convex payoff to intangible investment specifically, while physical investment remains subject to the standard real-options logic. The asymmetry across capital types, combined with the size-dependence of the exit option, produces the intangible gap: small firms near the exit margin tilt substantially toward intangible capital when σ_ω rises, while large firms with high continuation values are essentially unaffected.

Our paper is also closely related to the growing literature that documents and explains the importance and distinctiveness of intangible assets. [Eisfeldt and Papanikolaou \(2013\)](#) develop a model in which organizational capital as a production factor helps explain cross-sectional asset-pricing returns. [Ai, Croce, and Li \(2013\)](#) distinguish physical capital as assets in place from intangible capital as growth options in a quantitative asset-pricing framework, showing

that tangible and intangible capital can have different exposures to aggregate risk. [Gourio and Rudanko \(2014\)](#) study how customer capital—built through marketing and selling activities—shapes firm investment, profits, and Tobin’s q . [Sun and Xiaolan \(2019\)](#) provide a theoretical framework showing how firms use compensation contracts to finance investment in intangible assets. [Akcigit and Kerr \(2018\)](#) build a tractable growth model featuring internal and external innovation to account for firm size distributions. Our paper contributes by focusing on the riskiness of intangible accumulation and its implications for the cross-sectional variation in investment behavior.

2 Stylized Facts on Intangible Investments

2.1 Sample Construction

Our sample consists of firm-year observations from Compustat Fundamentals Annual over 1975–2023. Compustat provides detailed financial information on the publicly traded firms in the US. We restrict our sample to U.S.-incorporated firms. We exclude financial firms (SIC 6000 – 6999), regulated utilities (SIC 4900 – 4999), and firms categorized as public service, international affairs, or non-operating establishments (9000+). We also exclude firms with missing SIC code or with missing or negative values of sales, total assets, employees, R&D expenses, capital investment, current and long term liabilities and intangible assets. For details on data construction, see [Appendix A](#).

2.2 Measures of Intangible Capital

We measure intangible capital as the accumulated stock of intangible investment. The Compustat database provides detailed balance sheet and cash flow statement information that allows us to estimate its book value. We focus on a firm’s investment in its intellectual assets and innovation, as well as its organizational structure (including information technology and computer expenses) borrowing from the definition of intangible assets in [Brynjolfsson, Hitt, and Yang \(2002\)](#). According to the existing literature ([Corrado, Sichel, and Hulten \(2004\)](#), [Lev and](#)

Radhakrishnan (2005), Hulten and Hao (2008), Eisefeldt and Papanikolaou (2013), Falato et al. (2022), etc.), in the Compustat database, the computerized information (e.g., organizational structure) and economic competencies (i.e., knowledge embedded in firm-specific human and structural resources, including brand names) are recognized as a fraction of *Selling, General and Administrative* (SG&A) expenses, and the corresponding line for innovative property is *Research and Development* (R&D) expenses.

We follow Peters and Taylor (2017) to estimate the accumulated R&D capital and organizational capital using the perpetual inventory method.³ The stock of these capitals are defined as

$$H_{it} = (1 - \delta_h)H_{it-1} + I_{H,it}, \quad (1)$$

where H is the variable of interest, i represents firm indicator, t is the year, δ_h is depreciation rate, and I_H is the new investment on H . We also need to estimate the initial stock of H to keep track of the value over time, which is defined as

$$H_0 = \frac{I_0}{g_h + \delta_h},$$

where H_0 and I_0 are initial values, and g_h is the average annual growth rate. We use both SG&A expenditures and R&D expenses to construct our measure of intangible capital, H , for each firm year.

As an alternative measure of intangible capital, we adopt a market-based approach following (Kogan, Papanikolaou, Seru, and Stoffman, 2017), which infers the value of innovation from stock-market reactions to patent grant announcements. Specifically, we construct a firm-level dollar measure of intangible capital by aggregating the market-implied values of all patents granted to a firm in a given year. This measure captures investors' forward-looking assessments of the economic value of innovation and is available annually from 1926 to 2023.

Sun and Xiaolan (2019) and Eisefeldt, Falato, and Xiaolan (2023) proposed a compensation-based measure for the firm-level human capital, the total value of equity-based compensation. Since companies engaged in innovation and research-intensive activities require the input

³This is also used by the Bureau of Economic Analysis (BEA) to estimate R&D activities.

of high-skilled workers, and they often are paid with equity-based compensation, we adopt this measure to infer the human capital-related component of intangible capital from firms' expenses in their stock-based compensation. We use the cost of stock-based compensation (STKCO) and the value of shares reserved for equity-based compensation as our alternative compensation-based measure of firm-level intangible capital.

We also construct other financial and investment variables at the firm level throughout the entire sample. We obtain data on firms' IPO year to form firms' age based on their founding years. We construct Tobin's q as the ratio of the market value of total assets to the book value of total assets, ROA as the profits to total asset ratio. We use debt and equity flows to measure external finance following [Covas and Den Haan \(2011\)](#).

Table 1 reports the summary statistics of the main investment and financial variables by size groups. Firm size is measured by the market value of equity (ME). All variables are deflated using CPI. The statistics in Panel A reveal consistent patterns of firm dynamics that align with the existing literature. Younger firms tend to be smaller with higher growth and standard deviation in labor and total assets. Smaller firms also have higher Tobin's Q but lower profitability. The smallest firms (the bottom 20%), on average, have negative ROA .

Panel B reports investment dynamics across firm size groups and reveals a large and robust intangible investment gap. Across all measures, smaller firms exhibit substantially higher intangible intensity than larger firms. The stock of intangible capital—measured using R&D capital, organizational capital, or the intangible-to-physical capital ratio (PPENT)—is consistently highest among small firms, and this pattern is robust across alternative definitions. A market-based measure yields the same conclusion: patent value relative to total assets is greatest for small firms. Smaller firms also invest more aggressively in intangibles. Patent-related market value, employee stock-based compensation, R&D expenditures, and organizational capital investment (XSGA), each scaled by total assets, are all largest among small firms and decline monotonically with firm size. Advertising expenditures, which are included in selling, general, and administrative expenses, are unlikely to reflect innovation or research activity. Consistent with this interpretation, advertising expense ratios do not differ meaningfully across firm size groups.

Panel C reports measures of firms' financial conditions. Consistent with the existing literature, smaller firms face tighter financial constraints and a higher probability of entry and exit (Cooley and Quadrini, 2001). Across multiple measures, smaller firms in our sample have lower book and market leverage and are systematically more financially constrained than larger firms.

2.3 Stylized Facts

We summarize our findings on the time series behavior of intangible capital and physical capital across firm sizes. We document two central stylized facts. First, there is a large and rising intangible investment gap between small and large firms, driven by disproportionately higher intangible accumulation and investment among small firms rather than differences in physical investment. Second, intangible investment has become increasingly risky over time, with sharply rising volatility in both intangible capital accumulation and market-based valuations, particularly among smaller firms.

2.3.1 Intangible Investment Gap between Small and Large Firms

For each year, firms are classified into five groups based on their sales or market equity. Within each group, we calculate the median intangible-to-physical capital ratio ($hkratio$), where H is calculated using equation (1), and physical capital K is the value of *Property, Plant and Equipment* (PPEGT).

Figure 2 presents the time series of $hkratio$ by firm size. There exists substantial growth in intangible capital over time, consistent with Crouzet and Eberly (2023). The average intangible to physical capital ratio was lower than 1 before the 1980s and has risen to almost 2.8 in the 2000s. In the meantime, the growth in intangible capital behaves differently across size groups. We see much more rapid growth in intangible capital in the smallest firm size group. The difference in $hkratio$ between small and large firms represents the intangible gap. Since the 1980s, the accumulated investment in intangible capital has increased more than 6 times relative to the physical capital, with an increase of 7.1 times on average among the smallest firms. In con-

trast, the increase is only 2.2 times among the largest firms. This pattern is robust to alternative measures of intangible capital. Figure 3 decomposes the intangible-to-physical capital ratio into SG&A-based organizational capital and R&D-based innovation capital, showing that the intangible gap is present in both components.

The increase in the intangible gap is likely a result of the difference in investment activity over time. Figure 4 confirms that smaller firms invest more in intangible capital, i.e., R&D or organization capital (XSGA), than larger firms. Although this is true over the entire sample period, the gap of the intangible investment rate between the smallest size group and the largest size group widened, from 1.4% in 1980 to 15% in 2023. Intangible investment has therefore intensified more rapidly among smaller firms than among larger firms over time.⁴

Physical investment has remained broadly stable, with a slight downward trend since the 1980s. Figure 5 illustrates that physical capital investment rates move similarly across firm-size groups, with no meaningful divergence. This suggests that small firms differ from large firms in their *hkratio* primarily in their intangible investment, not in their physical investment activities.

We also want to highlight the distinction between accumulated intangible investment and balance sheet intangible assets. The balance sheet intangible assets (INTAN) reflect the overall value of acquired intangibles, including book values of different types of intangible assets: software, intellectual property, brand, business processes, and goodwill. Over the sample period, the intangible assets to physical capital ratio has been rising (Crouzet and Eberly, 2019) remarkably, but there is no substantial difference across size groups (Figure 6). In fact, larger firms experience faster growth in balance sheet intangible assets after the mid-1990s. One potential explanation is that intangible assets are outputs of intangible capital and results of within-firm innovation and acquisition activities. Larger firms accumulate balance-sheet intangible assets through acquisitions.

⁴One caveat is that our sample is a subset of the US economy since we are not able to examine the intangible investment activities in private firms. However, public firms contribute to more than 40% of US GDP and about one-third of the US employment. It represents a significant portion of the US economy. Importantly, private firms not in our sample are disproportionately small, young, and innovation-intensive—precisely the type of firms most likely to have high intangible-to-physical capital ratios. If anything, then, restricting attention to public firms is likely to understate the small-firm intangible gap we document.

2.3.2 Rising and Volatile Intangible Gap

The intangible gap documented in Section 2.3.1 has grown dramatically: since the 1980s, the intangible-to-physical capital ratio among the smallest firms has increased by more than 7 times, compared with only 2.2 times among the largest firms. What drives this widening dispersion? To understand the sources of the rise in aggregate intangible intensity, we decompose it using a cumulant expansion. This approach allows us to separate the contribution of the typical firm becoming more intangible-intensive from the contribution of increasing heterogeneity across firms. As we show below, the rise in aggregate intangible intensity is not simply a uniform shift: it is substantially driven by the growing cross-sectional dispersion in intangible capital ratios, reflecting the disproportionate intangible accumulation among small firms documented above.

We take a cumulant expansion approach to formally decompose the sources of the rise in aggregate intangible intensity. Let $R_{it} \equiv H_{it}/K_{it}$ denote firm i 's intangible-to-physical capital ratio in levels, and define $x_{it} \equiv \log R_{it}$. The object of interest is the cross-sectional average of the level ratio, $\bar{R}_t \equiv N_t^{-1} \sum_i R_{it}$, which captures the economically relevant aggregate intangible intensity. Because $\bar{R}_t = N_t^{-1} \sum_i e^{x_{it}}$, its logarithm equals the empirical cumulant generating function of x_{it} evaluated at $s = 1$:

$$\log \bar{R}_t = \underbrace{\kappa_1(t)}_{\text{Mean}} + \underbrace{\frac{\kappa_2(t)}{2}}_{\text{Variance}} + \underbrace{\frac{\kappa_3(t)}{6}}_{\text{Skewness}} + \underbrace{\frac{\kappa_4(t)}{24}}_{\text{Kurtosis}} + \dots, \quad (2)$$

where $\kappa_j(t)$ is the j -th cross-sectional cumulant of x_{it} . The first four cumulants coincide with familiar objects: κ_1 is the cross-sectional mean of x_{it} ; κ_2 is the variance; κ_3 is the third central moment (unnormalized skewness); $\kappa_4 = \mu_4 - 3\mu_2^2$ is the excess kurtosis contribution, which vanishes under normality. The decomposition is exact; we truncate at the fourth cumulant and collect the remainder in a residual term.

Figure 7 plots this decomposition over time. The mean component (κ_1) tracks the cross-sectional average of $x_{it} = \log(H_{it}/K_{it})$; it was negative through the mid-1990s—indicating that the typical firm had more physical than intangible capital—and has turned positive since 2000,

reflecting broad-based intangible deepening. The variance term ($\kappa_2/2!$) has grown steadily and now rivals the mean as the largest contributor to the aggregate intangible intensity \bar{R}_t . This means that a substantial and growing share of aggregate intangible intensity is driven not by the typical firm becoming more intangible-intensive, but by the widening cross-sectional dispersion in intangible capital ratios. The skewness contribution ($\kappa_3/3!$) has become increasingly negative over time, indicating a lengthening left tail in the cross-sectional distribution—a growing mass of firms with very low intangible intensity. This pattern suggests that while some firms have rapidly accumulated intangible capital, a substantial fraction of firms has been left behind, reinforcing the heterogeneity that amplifies aggregate intangible intensity through the variance channel. Together, these patterns confirm that the rise in aggregate \bar{R}_t is driven by both a shift in the typical firm’s intangible intensity and, critically, the growing heterogeneity across firms, consistent with the size-dependent intangible gap documented above.

What generates this rising cross-sectional dispersion? A natural candidate is the increasing volatility of intangible investment itself. Figure 8 plots the time series of firm-level *hkratio* volatility by size group, where volatility is measured as the standard deviation of *hkratio* over a rolling five-year window and then averaged within each size group. The figure shows a pronounced rise in intangible volatility among small firms—from 0.15 in 1980 to 2.55 in 2023—while volatility for large firms remains largely stable. This pattern indicates that intangible investment opportunities have become increasingly risky, particularly for small firms, and are notably more volatile than physical investment opportunities.

We also found similar patterns using the market-based measure. Figure 9 further illustrates this pattern using another data source on the value of innovation. The figure plots the standard deviation of the logarithm of normalized patent value, where each patent’s value is normalized by the median within the same CPC subclass and year to account for technological field heterogeneity, computed within a rolling three-year window. The cross-sectional dispersion in patent value rises sharply from the early 1980s onward, peaks around the early 2000s, and remains elevated thereafter. The increase in the volatility of patent values mirrors the rise in the volatility of intangible investment, as measured by *hkratio*, supporting the view that the riskiness of intangible capital has grown substantially over time. Taken together, the cumulant decomposi-

tion and the direct evidence on intangible volatility point to a common conclusion: the rising intangible gap is intimately linked to the increasing uncertainty of intangible investment, which disproportionately affects small firms and amplifies the cross-sectional dispersion in intangible intensity.

2.3.3 Financing Intangible Capital

A distinctive feature of intangible capital is that it can be financed internally through the labor market rather than through traditional debt and equity markets. Because intangible capital—particularly human and organizational capital—is largely non-pledgeable as collateral (Falato et al., 2022), firms cannot easily borrow against it. Instead, they finance intangible investment by deferring compensation and issuing equity stakes to employees, effectively allowing workers to act as “human capitalists” who bear part of the firm’s risk in exchange for claims on its future cash flows (Sun and Xiaolan, 2019; Eisfeldt, Falato, and Xiaolan, 2023; Guiso, Pistaferri, and Schivardi, 2012).

Figure 10 documents a steady rise in the aggregate share of corporate equity allocated to employee stock-based compensation over the past four decades, consistent with the secular increase in equity-financed compensation documented in Eisfeldt, Falato, and Xiaolan (2023). This financing channel is concentrated among smaller firms: as the cross-section in Table 1 shows, stock-based compensation as a share of total assets is 6.7 times larger for the smallest firms (5.69%) than for the largest (0.85%), a size gradient that closely mirrors the intangible gap documented in Section 2.3.1.

Taken together, these patterns suggest that equity-based financing of intangible capital has become both more prevalent over time and disproportionately concentrated among the smaller, intangible-intensive firms that drive the post-2000 widening of the H/K gap. This empirical regularity directly motivates the financing constraint we adopt in our quantitative model in Section 3.2: the firm’s effective borrowing capacity, $-\theta h_t$, scales with the stock of intangible capital, capturing the channel through which intangible firms relax financial frictions through equity-based compensation.

3 The Model

In this section, we propose an equilibrium model with firm dynamics to study firms' investment choices on intangible and physical capital over their life cycles. We begin by illustrating our main mechanism in a two-period environment and then extend the framework to a full quantitative model.

3.1 A Two-Period Model

There are two periods, $t = 1, 2$. At the beginning of the period $t = 1$, the firm owns intangible capital h_t and produces output $f(h_t) = h_t^\alpha$. The firm then decides on the intangible capital of the next period h_{t+1} . We assume the discount rate $\beta = 1$ and intangible capital does not depreciate each period ($\delta_h = 0$). We interpret h_t as the firm's size, since it is the only scale variable in this stripped-down setup; the dynamic model in Section 3.2 separates size from intangible composition by introducing physical capital and productivity.

Intangible capital faces uncertain productivity. The key assumption is that, at the beginning of the second period $t + 1$, intangible capital is subject to an intangible *quality* (IQ) shock ω_{t+1} that scales the productivity of the newly installed intangible capital. The effective capital level is $\omega_{t+1}h_{t+1}$. The firm then decides whether to operate or to exit after observing the quality shock ω_{t+1} . Without loss of generality, we assume that the quality shock ω_{t+1} is log-normally distributed, with $\ln(\omega_{t+1}) \sim N(\mu, \sigma^2)$.

We interpret the ω shock as the risk of investing in intangible capital. This risk includes the unexpected departures of firms' key talents, the failure of research and development projects, and the obsolescence of the existing organizational structure and knowledge. This shock is important for a few reasons. First, we interpret the shock as the changing quality of intangible capital, but it is unknown to all firms before the decision of investments made. The intangible quality shock affects the exit decision. As a result, investing in intangible capital is riskier for small and young firms since the quality shock may distort more on their incentives.

The firm operates only if the realized value is nonnegative, yielding the value

$$\max \{ (\omega_{t+1} h_{t+1})^\alpha - (h_{t+1} - h_t) - \xi, 0 \},$$

where $\xi > 0$ denotes a fixed operating cost.

The firm maximizes the value while making its investment decision:

$$V_t = \max_{h_{t+1}} E_t \left[\max \{ (\omega_{t+1} h_{t+1})^\alpha - K(h_{t+1}), 0 \} \right], \quad (3)$$

where $K(h_{t+1}) = (h_{t+1} - h_t) + \xi$. Assume the quality shock satisfies $\mu = -\frac{1}{2}\alpha\sigma^2$. This normalization implies $E[\omega_{t+1}^\alpha] = 1$. Analogous to the Black-Scholes-Merton framework, the firm's value is

$$V_t = \max_{h_{t+1}} h_{t+1}^\alpha N(d_1) - K(h_{t+1})N(d_2), \quad (4)$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution $N(0, 1)$, and $d_2 = \frac{\alpha \ln h_{t+1} - \ln K(h_{t+1}) - \frac{1}{2}\alpha^2\sigma^2}{\alpha\sigma}$, $d_1 = d_2 + \alpha\sigma$.

Proposition 1. *The optimal investment h_{t+1}^* is strictly increasing in the volatility of the quality shock σ .*

See Appendix B.1 for the proof.

The intuition is that the firm's payoff has a call-option-like structure and is convex in the quality shock. The firm's first-order condition for (3) $\alpha h_{t+1}^{\alpha-1} N(d_1) - N(d_2) = 0$ implies that the marginal benefit of investment is weighted by $N(d_1)$, which captures the upside potential, while the marginal cost is weighted by the survival probability $N(d_2)$. As σ rises, the asymmetry of the payoff structure—where losses are truncated at zero while potential gains are unbounded—becomes more pronounced. This leads to a relative increase in the weight on revenues ($N(d_1)$) compared to the weight on costs ($N(d_2)$). This widening 'wedge' effectively discounts the risk of investment. Consequently, the firm is incentivized to aggressively scale up h_{t+1} to capitalize on the increased likelihood of extreme positive realizations of the shock.

To further formalize this idea, we can write down the firm's problem where there is no option to exit. The firm's problem without option to exit reduces to $V_{0,t} = \max_{h_{t+1}} h_{t+1}^\alpha - K(h_{t+1})$ with

the optimal investment h_{t+1}^{**} characterized by $\alpha(h_{t+1}^{**})^{\alpha-1} - 1 = 0$. The value of the real option to exit is therefore given by $V_t - V_{0,t}$. The following proposition characterizes how this option value varies with the firm's initial size.

Proposition 2. *The value of the exit option is strictly decreasing in initial intangible capital h_t .*

See Appendix B.2 for the proof.

The value of the abandonment option declines as the firm becomes larger, as reflected in a higher stock of existing intangible capital h_t . An increase in h_t effectively pushes the firm deeper in the money: continuation becomes the dominant choice across almost all realizations of the quality shock, making the exit option unlikely to be exercised. As a result, the flexibility value embedded in the option to abandon diminishes with firm size.⁵

3.2 A Quantitative Model

Next we extend the two-period model to a dynamic equilibrium model with firm dynamics to study firms' investment choices on intangible and physical capital over their life-cycles. In this economy, firms make investment decisions in both intangible and physical capital with heterogeneous investment opportunity sets.

The Firm There is a continuum of incumbent firms at the beginning of the period t . The production requires the inputs of physical capital, k_t , intangible capital, h_t , and is subject to an idiosyncratic productivity shock, s_t , which follows AR(1) processes. The production function exhibits decreasing returns to scale,

$$F(s_t, k_t, h_t) = s_t k_t^{\alpha_k} h_t^{\alpha_h}. \quad (5)$$

Firms make investment decisions on physical capital, i_k , and intangible capital, i_h . The law

⁵In the two-period model, h_t enters the firm's value function V_t only through the investment cost $K(h_{t+1}) = (h_{t+1} - h_t) + \xi$, since there is no separate adjustment-cost term. Therefore Proposition 2 would be unchanged if h_t were relabeled as initial total assets. The result reflects a general size mechanism: a larger initial asset makes continuation more likely, lowering the exit threshold and the flexibility value of abandonment.

of motion of the physical capital takes the form of

$$k_{t+1} = (1 - \delta_k) k_t + i_{kt},$$

with a quadratic adjustment cost $g_k(i_{kt}) = \frac{\rho_k}{2} \left(\frac{i_{kt}}{k_t}\right)^2 k_t$, where ρ_k governs the magnitude of the adjustment cost.

Similarly, the law of motion of intangible capital takes the form of

$$\tilde{h}_{t+1} = (1 - \delta_h) h_t + i_{ht},$$

where i_{ht} is the total investment, and we assume a quadratic adjustment cost of capital, $g_h(i_{ht}) = \frac{\rho_h}{2} \left(\frac{i_{ht}}{h_t}\right)^2 h_t$, where ρ_h governs the magnitude of the adjustment cost.

Notice that \tilde{h}_t is not the effective intangible capital for the production function (5). At each period t , the quality of intangible capital is subject to some degree of uncertainty. Firms receive a random shock ω_t , immediately after observing s_t , and this shock determines the quality of their intangible capital for the production. The intangible stock, adjusted for its quality, evolves as follows.

$$h_t = \omega_t \tilde{h}_t, \tag{6}$$

where h_t is the amount of effective intangible capital used for production and \tilde{h}_t is the intangible capital at the beginning of period t . ω_t is unknown to all firms when the decision of exit and investment is made.

We assume that the intangible *quality* shock (IQ), ω , is independently and identically distributed (i.i.d.), and follows a log-normal distribution. Specifically,

$$\log \omega \sim N\left(-\frac{\sigma_\omega^2}{2}, \sigma_\omega^2\right),$$

which ensures that $\omega > 0$, almost surely and $\mathbf{E}[\omega] = 1$, $\text{Var}(\log \omega) = \sigma_\omega^2$.

The following figure 1 illustrates the within-period sequence of the firm's decisions, including the exit choice followed by the investment decision. Given the productivity shock s_t and

the quality shock ω_t , the physical capital stock k_t , and the intangible capital stock h_t , firms first decide whether to exit. If they choose to continue operating, they then make their investment decisions based on the observed productivity shocks and the effective intangible stock \tilde{h}_t .

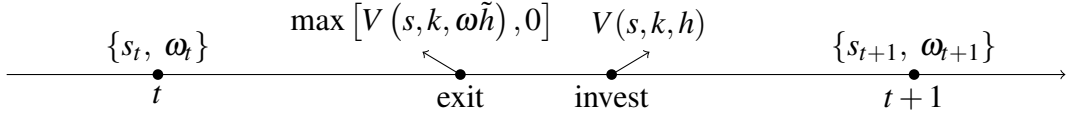


Figure 1: Timing of the Incumbent Firm

Financing Investment Firms have access to the financial market by raising funds to finance investments, but it is costly. Given the current capital stock k_t and h_t and the realized shocks, the firm chooses investment, i_{kt} and i_{ht} , and equity payout, d_t . A positive d_t means that distributions to shareholders, and a negative d_t denotes funds raised by the firm. The budget constraint is

$$i_{kt} + i_{ht} + d_t + g_h(i_{ht}) + g_k(i_{kt}) = F(s_t, k_t, h_t). \quad (7)$$

We assume that the firm's ability to raise funds is limited. We also abstract from explicitly modeling debt contracts and financial intermediaries in the benchmark model. Consistent with the existing literature (e.g., [Cooley and Quadrini \(2001\)](#)), small firms face tighter financing constraints, for instance, when using physical capital as collateral through debt contract.⁶ Intangible capital, however, can relax financing constraints through several channels.

First, because intangible capital is partly embodied in skilled workers and firm-specific human capital, firms can finance intangible investment by deferring compensation or by issuing equity-based compensation to employees. Such arrangements allow firms to substitute future claims for current wages and thereby free up internal funds for investment ([Babenko, Lemmon, and Tserlukevich, 2011](#); [Sun and Xiaolan, 2019](#); [Eisfeldt, Falato, and Xiaolan, 2023](#)). Second, some components of intangible capital, such as patents and other intellectual property, can

⁶Although small firms can in principle borrow, debt is costly ex ante due to default risk, so the marginal financing capacity from collateralized debt is small in our sample.

support external financing directly. More redeployable and more liquid patent assets increase lenders' expected recovery values and facilitate patent-backed venture lending (Hochberg, Serrano, and Ziedonis, 2018). Third, innovative and intangible-intensive firms may obtain financing from specialized equity investors, such as venture capital and private equity investors, who screen, monitor, and certify firms with risky intangible assets and can also improve access to follow-on financing (Hellmann and Puri, 2000; Hochberg, Serrano, and Ziedonis, 2018).

In the baseline, we set the pledgeability of physical capital to zero ($\theta_k = 0$), so that only intangible capital relaxes the financing constraint. This is a deliberate isolation, not a normalization: it cleanly separates the novel intangible-financing channel from the well-understood physical-capital collateral channel of Cooley and Quadrini (2001), and it matches the dominant financing margin observed in our sample (Section 2.3.3). Although the model does not distinguish among the three channels above, we summarize their combined effect by assuming that the net payout choice d_t satisfies:

$$d_t \geq -\theta h_t, \quad (8)$$

where $\theta > 0$ measures the effective financing capacity generated by intangible capital. This assumption implies that a higher level of intangible capital h_t expands the firm's ability to raise financing (i.e., to set $d_t < 0$), reflecting the mechanism that intangible capital improves the firm's financing capacity through the channels discussed above. As documented in Section 2.3.3, stock-based compensation as a share of total assets is roughly 6 times larger for the smallest firms than for the largest, with a cross-sectional size gradient that closely mirrors the intangible gap—directly motivating the proportionality between financing capacity and the intangible-capital stock h_t in equation (8). Section 4.3 returns to the joint case ($\theta_h, \theta_k > 0$). Under empirically reasonable parameterization, the H/K amplification mechanism is robust to allowing physical-capital pledgeability.

Exit Each period, the firm makes exit decisions. The firm exits the market if the continuation value, conditional on the realization of idiosyncratic productivity shock s_t and intangible quality shock ω_t , is negative. All incumbent firms have to pay a fixed cost of production of ξ . Incumbent firms produce consumption goods, make investment decisions on both physical

and intangible capital, and choose new equity levels when maximizing the value of the firm $V(s, k, h)$:

$$V(s_t, k_t, h_t) = \max_{d_t, i_{kt}, i_{ht}} d_t - \xi + \beta \mathbf{E}\{\max[V(s_{t+1}, k_{t+1}, \omega_{t+1} \tilde{h}_{t+1}), 0]\}, \quad (9)$$

subject to the firm's budget constraint (7) and financing constraint (8).

Given productivity shock s_t , physical capital stock k_t , and the intangible capital stock before the quality shock \tilde{h}_t , the exit decision involves a reservation rule

$$\eta_t^e(s_t, k_t, \tilde{h}_t) = \begin{cases} 0 & \text{if } \omega \geq \omega^*(s_t, k_t, \tilde{h}_t) \\ 1 & \text{o.w.} \end{cases},$$

where

$$\omega^*(s_t, k_t, \tilde{h}_t) = \inf\{\omega \in \Omega : V(s_t, k_t, \omega \tilde{h}_t) \geq 0\}.$$

Entry At the beginning of each period, a total mass of N potential entrants randomly draws an initial idiosyncratic productivity, s_t^0 , from the distribution with CDF $\Gamma(s_t^0)$. After drawing the s_t^0 , a potential entrant decides whether to enter the market. If it enters, it pays the entry cost c_e . The new entrant maximizes the expected value of the firm, $V^e(s_t^0)$, by choosing labor and capital inputs and equity to fund investment

$$V^e(s_t^0) = \max_{d_t, k_t, h_t} d_t + \beta \mathbf{E}\{\max[V(s_{t+1}, (1 - \delta_k)k_t, \omega_{t+1}(1 - \delta_h)h_t), 0]\}$$

subject to the budget constraint (7) and financing constraint (8).

At time $t + 1$, an entrant will face exactly the same problem as the incumbent. Thus, the firm enters the market if and only if the discounted value of entering exceeds the entry cost

$$V^e(s_t^0) \geq c_e.$$

This equation determines the threshold, s_e^* , beyond which only firms with idiosyncratic productivity shocks, $s_t^0 \geq s_e^*$, enter the market.

Recursive Equilibrium We consider a recursive equilibrium definition. A key element here is the law of the motion of firm size distribution. (μ_t) are the aggregate state variables in our model. We call μ_t the aggregate state of the industry, which depends on the current measurements of firms physical and intangible stocks, and idiosyncratic productivity shocks.

Given firms' entry, exit, and default decisions and policy functions, the evolution of the state of the industry μ satisfies

$$\begin{aligned} & \mu(s', k', \tilde{h}') \\ &= \int \left\{ p^i(s'|s) \int_{\omega \geq \omega^*(\mu)} 1_{(k', \tilde{h}'|k, \omega \tilde{h}, s)} \mu(s, k, \tilde{h}) \right\} H^\omega(d\omega) ds \\ & \quad + N \int_{s \geq s^*(\mu)} 1_{(k, \tilde{h}')} p^i(s'|s) \Gamma(ds), \end{aligned} \tag{10}$$

where p^i is the conditional probability density functions of idiosyncratic productivity and aggregate shocks, respectively; H^ω is the cumulative distribution function of capital shocks; and $1_{(k, \tilde{h}'|k, \omega \tilde{h}, s)}$ and $1_{(k, \tilde{h}')}$ are the indicator functions given incumbents' and entrants' policy functions

$$\begin{aligned} 1_{(k', \tilde{h}'|k, \omega \tilde{h}, s)} &= 1_{\{\zeta_k^i(k, \omega \tilde{h}, s)=k', \zeta_h^i(k, \omega \tilde{h}, s)=\tilde{h}'\}}, \\ 1_{(k', \tilde{h}')} &= 1_{\{\zeta_k^e(s)=k', \zeta_h^e(s)=\tilde{h}'\}}, \end{aligned}$$

where ζ_k^i, ζ_h^i are the policy functions of incumbents, and ζ_k^e, ζ_h^e are the policy functions of entrants.

A recursive competitive equilibrium consists of the exit decision $\eta^e(s, k, \tilde{h})$, the entry decision $s_e^*(s)$, the optimal decision rules, and the measure of entrants N_t such that incumbents' optimization and entrants' optimization are solved.

The economy is characterized by a certain distribution of firms, μ , over all state variables, k, h and s . In the analysis described in this section, we focus on the invariant distributions of firms. The existence and uniqueness of the invariant distribution therefore depends on the properties of the transition matrix generated by the optimal policy functions, physical and intangible capital, of both incumbents and entrants. The invariant distribution is the fixed point

in this contraction mapping.

Proposition 3. *An invariant measure of firms, μ^* , exists. μ^* is unique, and the sequence of measures generated by the transition functions $\Psi, \{\Psi^n(\mu_0)\}_{n=0}^\infty$, converges weakly to μ^* from any initial arbitrary μ_0 .*

See Appendix B.3 for the proof.

3.3 Investment Wedge: Real Option and Financing Channels

We now analyze the firm's intangible and physical capital investment decisions. We can write down the investment Euler equation for k and h :

$$(1 + \lambda_t)q_{k,t} = \beta \mathbf{E}_t \left[\left((1 + \lambda_{t+1}) \left(\alpha_k \frac{F_{t+1}}{k_{t+1}} + (1 - \delta_k)q_{k,t+1} + \frac{\rho_k}{2} \left(\frac{i_{k,t+1}}{k_{t+1}} \right)^2 \right) \right) \cdot \mathbf{1}_{\{\omega_{t+1} > \omega_{t+1}^*\}} \right], \quad (11)$$

$$(1 + \lambda_t)q_{h,t} = \beta \mathbf{E}_t \left[\omega_{t+1} \cdot \left((1 + \lambda_{t+1}) \left(\alpha_h \frac{F_{t+1}}{h_{t+1}} + (1 - \delta_h)q_{h,t+1} + \frac{\rho_h}{2} \left(\frac{i_{h,t+1}}{h_{t+1}} \right)^2 \right) + \lambda_{t+1} \theta \right) \cdot \mathbf{1}_{\{\omega_{t+1} > \omega_{t+1}^*\}} \right], \quad (12)$$

where $q_{k,t}$ and $q_{h,t}$ are the marginal cost of investment, and λ_t is the Lagrange multiplier associated with the financial constraint $d_t \geq -\theta h_t$. The left-hand side of each Euler equation captures the marginal cost of purchasing and adjusting capital, while the right-hand side represents the expected marginal benefit of an additional unit of investment at time t . In the absence of intangible-quality risk ω_t and when financial frictions do not bind $\lambda_t = 0$, the two Euler equations are symmetric in k and h , differing only through the production elasticities α_k, α_h and the depreciation and adjustment-cost parameters. The optimal policies imply that the intangible-to-physical capital ratio H'/K' is pinned down by these technology parameters and is *independent of firm size*. Hence, the benchmark model cannot generate the size-dependent intangible gap documented in the data—doing so would require imposing exogenous heterogeneity in firm

composition or technology.⁷

Exit Option A key distinction between physical and intangible investment emerges once we introduce the quality shock ω_{t+1} and the endogenous exit option. With these features, the Euler equation for physical capital retains the standard structure: the marginal investment cost $(1 + \lambda_t)q_{k,t}$ equals the expected discounted marginal return, conditional on firm survival $\omega_{t+1} > \omega_{t+1}^*$. In contrast, the marginal return to intangible investment includes an additional multiplicative term ω_{t+1} , which makes the payoff to intangible capital strictly convex in the quality realization. This convexity creates a real option: the firm incurs investment costs today but retains the ability to exit tomorrow if realized quality is sufficiently low. The survival indicator $\mathbf{1}_{\{\omega_{t+1} > \omega_{t+1}^*\}}$ truncates low-quality states, allowing the firm to walk away from unfavorable outcomes while disproportionately benefiting from high-quality realizations amplified by ω_{t+1} . Large firms rarely exit, so their continuation value is almost always positive and the abandonment option contributes little to their Euler equation. Small firms, whose continuation values lie close to zero, face a much tighter exit threshold. For them, intangible investment resembles a one-sided bet: downside losses are eliminated through exit, while upside gains scale with ω_{t+1} . Consequently, the real option embedded in intangible capital makes such investment particularly attractive for young and small firms.

The magnitude of this option value is increasing in the volatility of the quality shock σ_ω . When $\log \omega_{t+1} \sim N(\mu_\omega, \sigma_\omega^2)$, multiplying the marginal return of intangible investment by ω_{t+1} is equivalent to evaluating the return under a tilted lognormal distribution $\tilde{\omega}_{t+1}$ whose log-mean is shifted to the right by exactly σ_ω^2 .⁸ A higher σ_ω therefore disproportionately increases the weight placed on high-quality realizations, raising the expected marginal return to intangible

⁷Even if sensitivities differ modestly, one would need additional assumptions—such as a changing composition toward intangible-intensive small firms—to match the empirical patterns.

⁸A useful property of the lognormal distribution helps interpret how the quality shock affects the marginal return to intangible capital in the Euler equation. Suppose $\log \omega \sim N(\mu_\omega, \sigma_\omega^2)$ so that $E[\omega] = 1$. For any measurable function $g(\omega)$, the expectation weighted by ω can be written as

$$E[\omega g(\omega)] = \int \omega g(\omega) f(\omega) d\omega.$$

Since $\omega f(\omega)$ is itself a probability density under the normalization $E[\omega] = 1$, tilting by ω is equivalent to evaluating $g(\cdot)$ under a shifted lognormal distribution. Specifically, $\omega f(\omega)$ corresponds to a lognormal distribution

investment. This mechanism explains why the value of the real option embedded in intangible capital is strictly increasing in the dispersion of the shock: uncertainty expands the upside without changing the downside, because the downside is endogenously cut off by the exit threshold. The policy functions plotted in Figures 11 illustrate these mechanisms quantitatively. Figure 11 compares the optimal intangible-to-physical capital ratio under low and high quality-shock volatility in the absence of financial frictions. The figure shows that in the environment of large intangible uncertainty (σ_ω), the H'/K' ratio is much larger for smaller firms (measured as low k or low h).

To quantify the magnitude of the exit option in the dynamic model, define the *exit option value* as the gain from the firm's ability to walk away when $\omega < \omega^*$: instead of a negative continuation value, the firm collects zero, and the avoided loss is the option's contribution to firm value. Figure 13 plots this contribution along the diagonal $K = H$ at the median productivity level, reported as the percentage change in firm value $(V^{\text{exit}} - V^{\text{no exit}})/V^{\text{no exit}}$, where V^{exit} and $V^{\text{no exit}}$ denote the value functions with and without the exit option. The exit option value is largest for small firms, whose continuation values lie close to zero and for whom the exit threshold ω^* is high; it vanishes as firm size increases, since large firms continue almost surely. This confirms, within the full dynamic environment, the mechanism formalized in Proposition 2 of the two-period model: the real-option value embedded in intangible investment is concentrated among small firms near the exit margin, which is precisely the channel through which higher quality-shock volatility σ_ω tilts these firms more aggressively into intangible capital.

Financing Channels Financial frictions further strengthen the attractiveness of intangible capital. When the financing constraint binds ($\lambda_{t+1} > 0$), the firm attaches a shadow value to internal liquidity. Intangible capital relaxes the financing constraint through θ , as reflected in the

whose log-mean shifts from μ_ω to $\mu_\omega + \sigma_\omega^2$. Hence,

$$E[\omega g(\omega)] = E_{\ln \tilde{\omega} \sim N(\mu_\omega + \sigma_\omega^2, \sigma_\omega^2)}[g(\tilde{\omega})].$$

Intuitively, the tilted distribution has the same log-variance σ_ω^2 as the original but a log-mean shifted to the right by exactly σ_ω^2 . As σ_ω increases, two reinforcing effects operate: the location of the tilted distribution moves further to the right, and its dispersion widens. Both effects place greater weight on high-quality realizations, amplifying the convex option value embedded in intangible investment.

additional term $\lambda_{t+1}\theta$ in its Euler equation. This creates a complementarity between the real option and financial frictions: constrained firms benefit more from states where the firm survives and receives a high-quality intangible realization, because those states not only generate higher marginal product but also loosen the financing constraint going forward. Consequently, constrained firms place an even higher value on intangible capital relative to physical capital, amplifying the endogenous shift toward intangible investment among small and financially constrained firms. Figure 12a demonstrates that when financial frictions are present, the same increase in volatility substantially elevates the H'/K' ratio over a wider range of firm sizes, particularly for small firms near the exit margin.

It is worth noting that the assumption that intangible capital plays a unique role in relaxing firms' financial constraints may appear to favor younger firms' tilt toward intangible investment. Although this assumption is grounded in empirical evidence, the model's qualitative and quantitative results still hold even if physical capital can also be financed through the internal labor market. The key mechanism is that younger firms benefit from the real option embedded in the quality shock; however, if they are financially constrained, one would not expect large movements in intangible investment simply because of limited resources. It is the combination of a less stringent financial constraint and the option value associated with intangible capital that generates the observed intangible investment gap in the data. Figure 12b confirms that the two channels are complementary, not redundant: setting $\theta = 0.01$ —so that intangible capital provides essentially no borrowing capacity—substantially compresses the volatility-driven increase in H'/K' across all sizes, even though the real-option mechanism (operating through the exit threshold) still applies. This shows that the size-dependent intangible gap requires both the option mechanism and the financing channel to be active; either alone is insufficient to match the cross-sectional spread observed in the data.

4 Calibration and Quantitative Analysis

In this section, we bring the model to the data. We calibrate the quantitative model with the intangible quality shock introduced in Section 3.2, and then quantify how intangible capital

uncertainty, together with financial frictions, shapes intangible intensity across the firm-size distribution.

4.1 Calibration

We now describe our choices of parameter values, which we report in Table 3. We set $\beta = 0.96$, which corresponds to an annual discount rate of roughly 4%. $\alpha_h = 0.4$ and $\alpha_k = 0.3$ are chosen to match the median H/K ratio of 1.36 in the data over the period 1975–2000. To isolate the impact of the volatility of the intangible quality shock, we impose the same depreciation rate and adjustment cost for physical and intangible capital, setting $\delta_h = \delta_k = 0.1$, $\rho_h = \rho_k = 0.1$. The investment adjustment cost and depreciation rate are chosen to match the investment rate of physical and intangible capital in the data. We set the fixed production cost to $\xi = 2.45$, which helps match the targeted exit rate for U.S. firms. The idiosyncratic productivity shock follows an AR(1) process in logs with persistence parameter ρ_s and standard deviation of innovations σ_s . We set $\rho_s = 0.8$ and $\sigma_s = 0.1$, similar to models in the literature on investment with firm-level productivity shocks (see [Hennessy and Whited \(2005\)](#)).

We calibrate the benchmark model to U.S. firm-level data over the period 1975–2000. This pre-shift sample serves as our baseline because both of our main channels—the quality-shock volatility σ_ω and the intangible-financing capacity θ —shifted materially after 2000: the cross-sectional dispersion of intangible quality rose sharply (Section 2.3.2), and the share of corporate equity allocated to employee compensation roughly doubled (Section 2.3.3). Calibrating to the early period yields a benchmark whose remaining parameters are uncontaminated by these structural shifts, allowing us to quantify in Section 4.2 how much of the post-2000 widening in the intangible gap is attributable to movements in σ_ω and θ alone, holding all other parameters fixed at their 1975–2000 values.

We calibrate the key parameters governing intangible investment frictions, the standard deviation of the quality shock, σ_ω , and the intangible capital financing capacity, θ , using cross-sectional moments that directly capture the dispersion in intangible investment across firm sizes. Specifically, we target the H/K spread, defined as the difference in the intangible-to-physical capital ratio between firms in the smallest and largest size quintiles, as well as

the employee-equity-to-market-equity ratio, which reflects the extent to which firms rely on equity-based compensation to fund intangible investment. These two moments jointly calibrate σ_ω and θ : a higher σ_ω increases the option value of intangible investment and amplifies the H/K spread, while a higher θ relaxes intangible financing constraints and raises the share of employee equity in firm value.

Table 4 reports the model’s fit relative to these empirical moments. The benchmark moments are calculated from the sample period 1975–2000. Panel A compares the model moments under $(\theta = 0.8, \sigma_\omega = 0.2)$ with their data counterparts, while Panel B evaluates the size distribution of intangible intensity, showing that the model successfully reproduces the cross-sectional pattern of H/K ratios across firm-size groups. The spread in the H/K ratio between the bottom 15% and the top 75% of firms is 0.467 in the data, and the model generates a comparable gap of 0.50 while also closely matching the level of the H/K ratio within each size quintile.

4.2 Explaining the Post-2000 Intangible Gap

Given the sharp rise in the uncertainty of intangible investment and output documented in Section 2.3.2, we ask how much of the increase in the intangible gap since 2000 can be attributed to the rise in the quality shock.

We begin with the benchmark calibration $(\theta = 0.8, \sigma_\omega = 0.2)$ reported in Table 4, which matches key cross-sectional moments in the data before 2000. The benchmark successfully reproduces the pre-2000 H/K spread, the share of employee equity in firm value, average investment rates, and the average firm exit rate. Empirically, the intangible gap increased to 1.374—nearly tripling since 2001—and the quality-uncertainty parameter σ_ω rose by 37.5%, consistent with the increase in the normalized standard deviation of patent values documented in Figure 9.

Table 5 reports comparative statics that vary σ_ω while holding θ fixed. As the volatility parameter increases from 0.2 to 0.275, the intangible gap expands. A higher σ_ω amplifies the real-option value of intangible investment, disproportionately raising intangible intensity among small firms and widening the dispersion in H/K ratios across size groups. Quantita-

tively, increasing σ_ω generates a substantial rise in the intangible gap. The model-implied H/K spread increases from 0.503 in the benchmark to 0.812 under the higher-volatility calibration. This response accounts for roughly 60% of the observed post-2000 H/K spread of 1.374 in the data (2001–2023). Importantly, the rise in the intangible gap is not driven by an increase in θ . Instead, the results imply that higher quality-shock volatility can account for a large fraction of the observed rise in intangible investment concentration, even without additional easing of financial constraints.

Increasing θ does not necessarily widen the cross-sectional H/K spread. A higher θ raises the overall financing capacity of intangible-intensive firms, and moves small firms away from both the constrained margin and the exit region. As small firms become less likely to exit, the option value embedded in risky intangible investment declines. Moreover, larger firms also benefit from the relaxation of financing constraints and can increase their intangible investment. This explains why the H/K spread falls when θ increases from 0.8 to 1.2 in Table 5, holding σ_ω fixed.

4.3 Model Extension: Generalized Financial Frictions

In this section, we consider a more general specification of financial frictions that allows both intangible and physical capital to relax borrowing constraints. Specifically, the firm can borrow up to some capacity of the amount of capital the firm can pledge. The financial constraint is now written as:

$$d_t \geq -(\theta_h h_t + \theta_k k_t), \quad (13)$$

where θ_h and θ_k govern the pledgeability of intangible and physical capital, respectively. By allowing both physical and intangible capital to serve as collateral, firms can borrow up to θ_k times their physical capital in addition to θ_h times their intangible capital when payouts are negative.

Specifying physical capital as a relaxer of the payout floor on the same terms as intangible capital—i.e., $\theta_k k_t$ entering the constraint $d_t \geq -\theta_h h_t - \theta_k k_t$ with the same equity-like functional form—is a generous assumption, and it is worth being explicit about why. In the

canonical collateral-constraint literature (Kiyotaki and Moore, 1997; Bernanke, Gertler, and Gilchrist, 1999; Albuquerque and Hopenhayn, 2004; Rampini and Viswanathan, 2013; Cooley and Quadrini, 2001), physical capital is pledgeable to *external* lenders through debt contracts that carry default and intermediation costs; the borrowing capacity $\theta_k k_t$ in those models maps to debt rather than to the internal, equity-like financing modeled in our baseline. Empirically, Lian and Ma (2021) show that the bulk of U.S. corporate borrowing is cash-flow-based rather than asset-based, especially among the small, intangible-intensive firms that drive our cross-sectional gap; Falato et al. (2022) reinforce the asymmetry from the opposite direction, documenting that as firms become more intangible-intensive, debt capacity tied to physical-capital pledgeability *falls* rather than rises. The internal financing channel underlying our baseline θ_h —deferred compensation and equity-based pay to skilled workers (Sun and Xiaolan, 2019; Eislefeldt, Falato, and Xiaolan, 2023)—has no analog for physical capital: tangible assets do not generate borrowing capacity through the labor market. The natural empirical asymmetry is therefore $\theta_h > \theta_k$, with θ_k further dampened by the default and intermediation costs that the constraint above abstracts from. We adopt the symmetric specification here as a deliberate stress test: if physical capital can also relax the payout floor frictionlessly, we are giving the alternative channel the most favorable shot.

To study how the collateral role of physical capital affects firms’ capital composition, Figure 14 plots the effect of θ_k on the cross-sectional dispersion in intangible intensity, measured by the H/K spread between small firms (below the 15th percentile) and large firms (above the 75th percentile of size) while fixing $\theta_h = 0.8$. We compare an economy with uncertain quality shocks ($\sigma_\omega = 0.275$, blue line) to an otherwise identical economy without volatility shocks (yellow line). Based on the empirical evidence in Lian and Ma (2021), the plausible range for θ_k is approximately $[0.05, 0.10]$, far below our benchmark $\theta_h = 0.8$.⁹ Both estimates are

⁹Two pieces of evidence in Lian and Ma (2021) bound θ_k . (i) *Direct anchor*. Their Table 8 reports that asset-based lending averages approximately 2% of total assets in the U.S. corporate sample. Equating the model’s physical-capital financing capacity $\theta_k k_t$ with observed asset-based lending and using a typical ratio of total assets to physical capital of about 2.4 in our sample (since total assets $\approx k + h$ with $h/k \approx 1.36$), this implies $\theta_k \approx 0.02 \times 2.4 \approx 0.05$. (ii) *Pledgeability haircut*. Asset-based debt is collateralized at *liquidation* value rather than *going-concern* value: Lian and Ma (2021) document advance rates of approximately 60% on the book value of eligible inventory, and liquidation values for fixed assets are typically 30–60% of book value, giving an effective borrowing capacity per dollar of physical-capital book value of at most $0.60 \times 0.50 = 0.30$.

upper bounds in the sense that they abstract from the default and intermediation costs of asset-based debt that the model's symmetric constraint does not capture; the equity-equivalent θ_k is smaller still. Reading off Figure 14 at this range of θ_k , the model produces an H/K spread of approximately 0.3 to 0.35. Even with this generous setup, our estimate only weakens the result modestly.

The figure also highlights the interaction between uncertainty, financial frictions, and firm size. With decreasing returns to scale, the production technology implies concavity of the value function, so quality volatility has two opposing effects. On the one hand, uncertainty generates positive real option value by increasing the value of waiting and selective investment. On the other hand, concavity implies an uncertainty loss: volatility reduces expected profits through Jensen's inequality. These forces operate differently across firms. For smaller firms, borrowing constraints are more likely to bind, which raises the marginal value of relaxing financial frictions through investment. As a result, both the option value of uncertainty and the associated uncertainty loss are amplified for constrained firms.

Consistent with this intuition, when θ_k is small—so that physical capital provides limited collateral value—the presence of quality volatility increases the H/K spread relative to the no-uncertainty benchmark. In this region, the option value effect dominates the uncertainty loss, leading small firms to tilt more strongly toward intangible investment. However, as θ_k increases and physical capital becomes more effective at relaxing financial constraints, the incremental benefit of intangible investment declines, especially for constrained firms. In this case, the concavity-induced uncertainty loss dominates, and the H/K spread becomes larger in the economy without volatility shocks.

Overall, these results show that the effect of uncertainty on firms' capital composition crucially depends on the collateral role of physical capital, and that our baseline findings are robust to allowing for a more general form of financial frictions.

5 Conclusions

We document a persistently rising intangible-investment gap between small and large firms: small firms allocate a substantially higher share of resources to intangible relative to physical capital, even though they are typically more financially constrained. To understand the mechanisms consistent with these heterogeneous patterns, we develop a dynamic industry-equilibrium model featuring both intangible and physical capital. The key mechanism is the option-like nature of intangible investment, which, combined with the rising uncertainty of intangible outcomes, generates the observed divergence. Our quantitative analysis shows that the model accounts for approximately 60% of the post-2000 increase in the intangible gap. This highlights that the rise in intangible intensity cannot be explained solely by changes in the aggregate production function.

Our findings underscore a broader transformation in the U.S. corporate sector: intangible investment is increasingly driven by smaller, younger, and lower-valuation firms. These firms carry much of the economy's innovation risk, conduct more intangible investment, and disproportionately shape the cross-sectional dispersion in investment behavior. Understanding the dynamics of intangible accumulation—its risks, financing constraints, and heterogeneous adoption across firm types—is therefore essential for interpreting modern corporate investment patterns, evaluating innovation policy, and assessing the long-run implications of the intangible economy.

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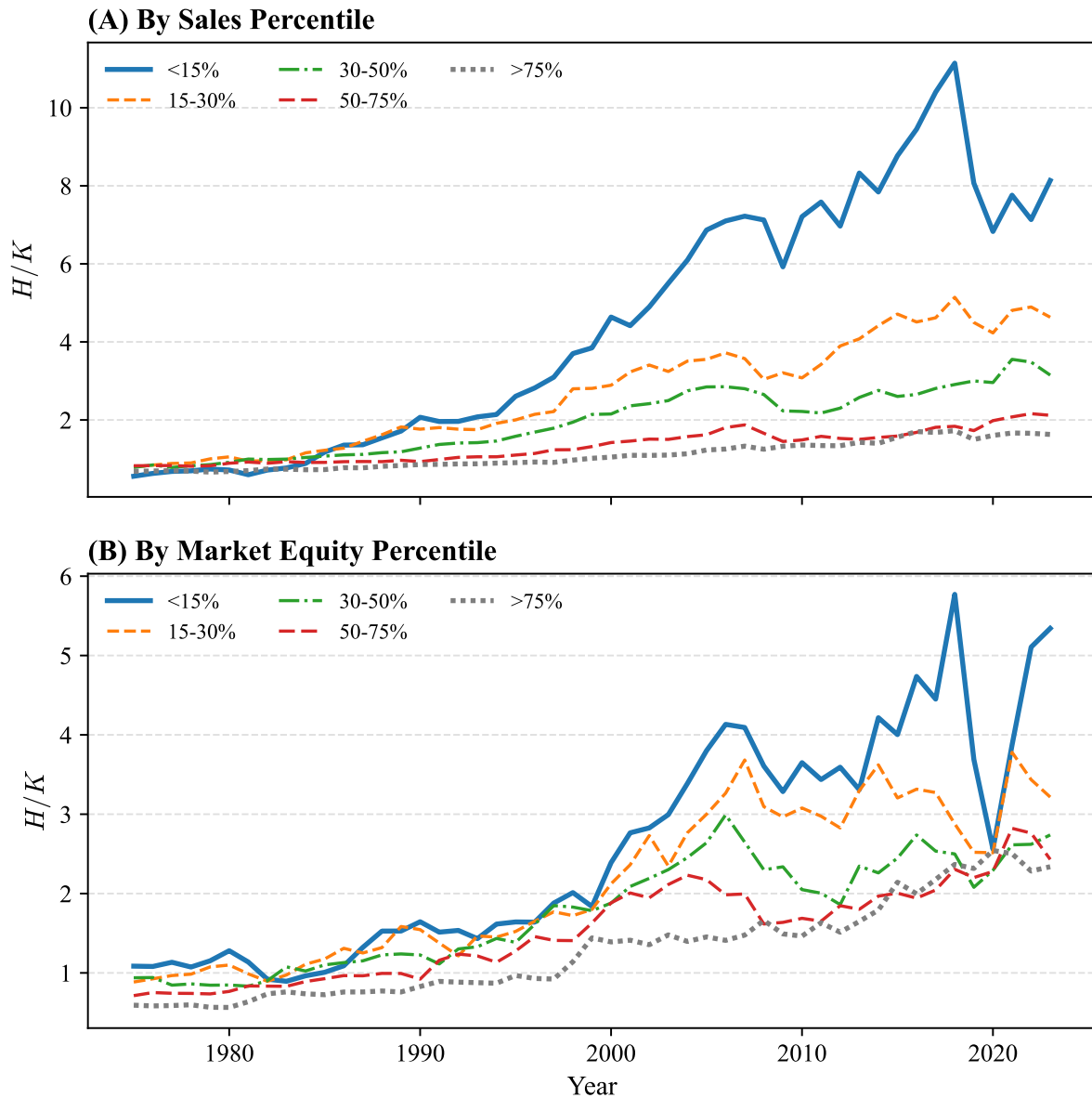


Figure 2: Intangible-Physical Capital Ratio by Firm Size. This figure plots the intangible (H) to physical (K) capital ratio (hkratio) by size over time. Firm size is measured by either sales or market equity, and the hkratio is measured as accumulated intangible investment following Peters and Taylor (2017), H, divided by PPEGT. For each year, firms are sorted into five groups based on their sales or market equity, and the median hkratio is computed within each group to mitigate the impact of outliers.

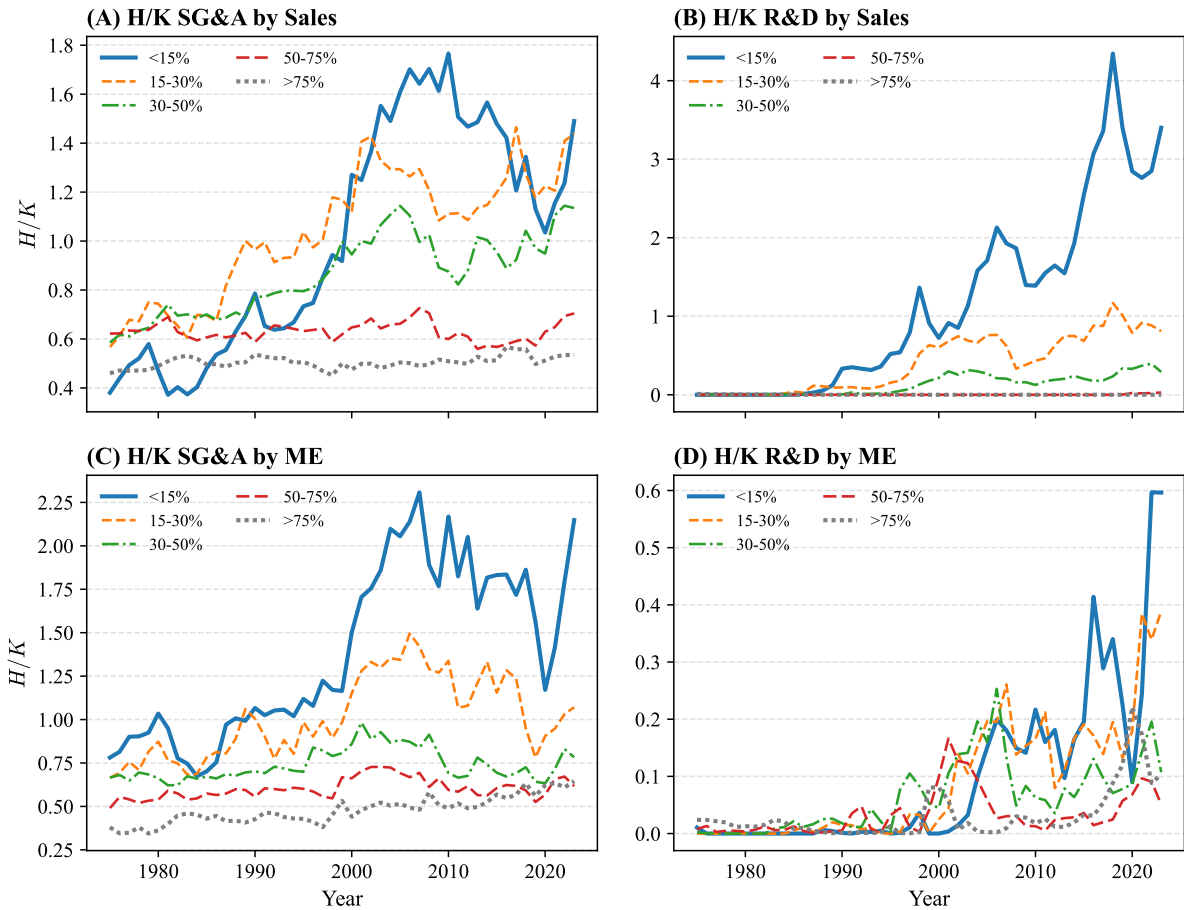


Figure 3: Intangible-Physical Capital Ratio by Firm Size (Robustness). This figure plots additional two types of intangible-to-physical capital ratios. The top panels show ratios constructed from SG&A-based organizational capital (Panel A and Panel C) and R&D-based innovation capital (Panel B and Panel D) across sales percentile groups. The bottom panels present the corresponding ratios across market equity percentile groups. For each year, firms are sorted into five groups based on their sales or market equity, and the median ratio of intangible capital (H) to tangible capital (K, measured by PPEGT) is computed within each group to mitigate the impact of outliers.

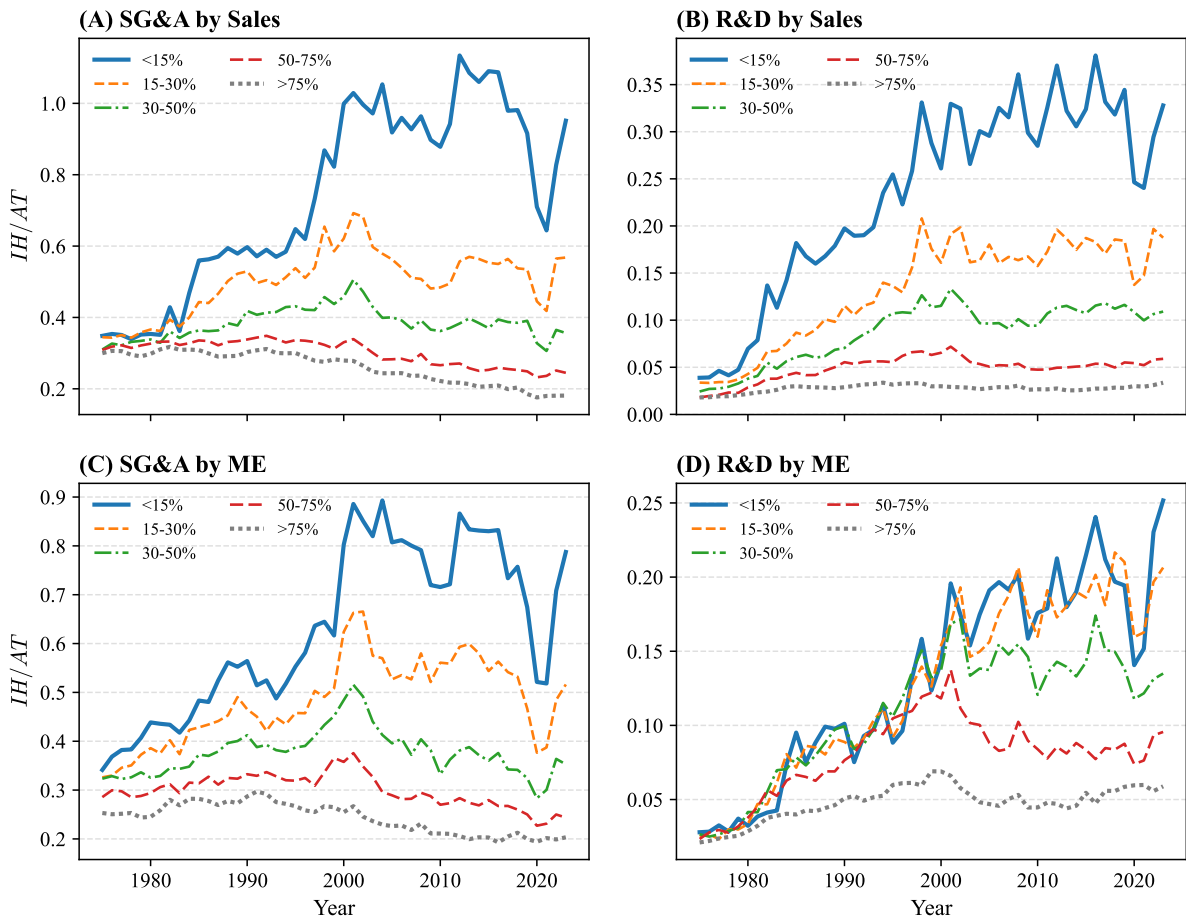


Figure 4: Investment Rate of Intangible Capital by Firm Size. This figure presents the investment in intangible capital (ih) to total assets ratio (AT) by size over time. Size is measured by sales or market equity, and the investment in intangible capital is measured as XRD or XSGA in Compustat. For each year, firms are categorized into five groups based on their sizes, and we estimate the average ratio within each group for a given year. Source: Compustat Fundamentals Annual for 1975-2023.

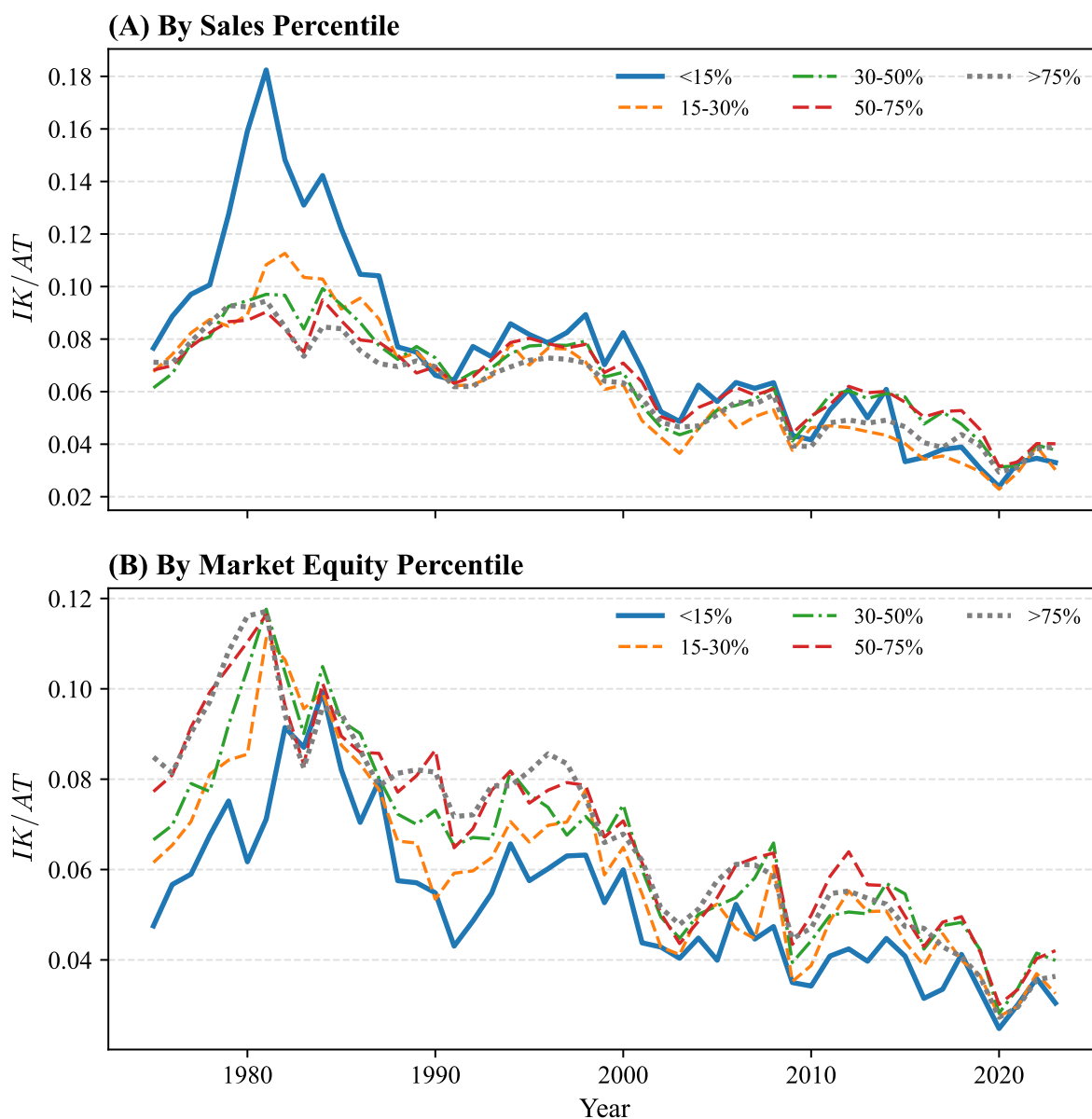


Figure 5: Investment Rate of Physical Capital by Firm Size. This figure presents the investment in physical capital (K) to total assets ratio (ik) by size over time. Size is measured by sales or market equity, and the investment in physical capital is measured as CAPX in Compustat. For each year, firms are categorized into five groups based on their sizes, and we estimate the average ratio within each group for a given year. Source: Compustat Fundamentals Annual for 1975-2023.

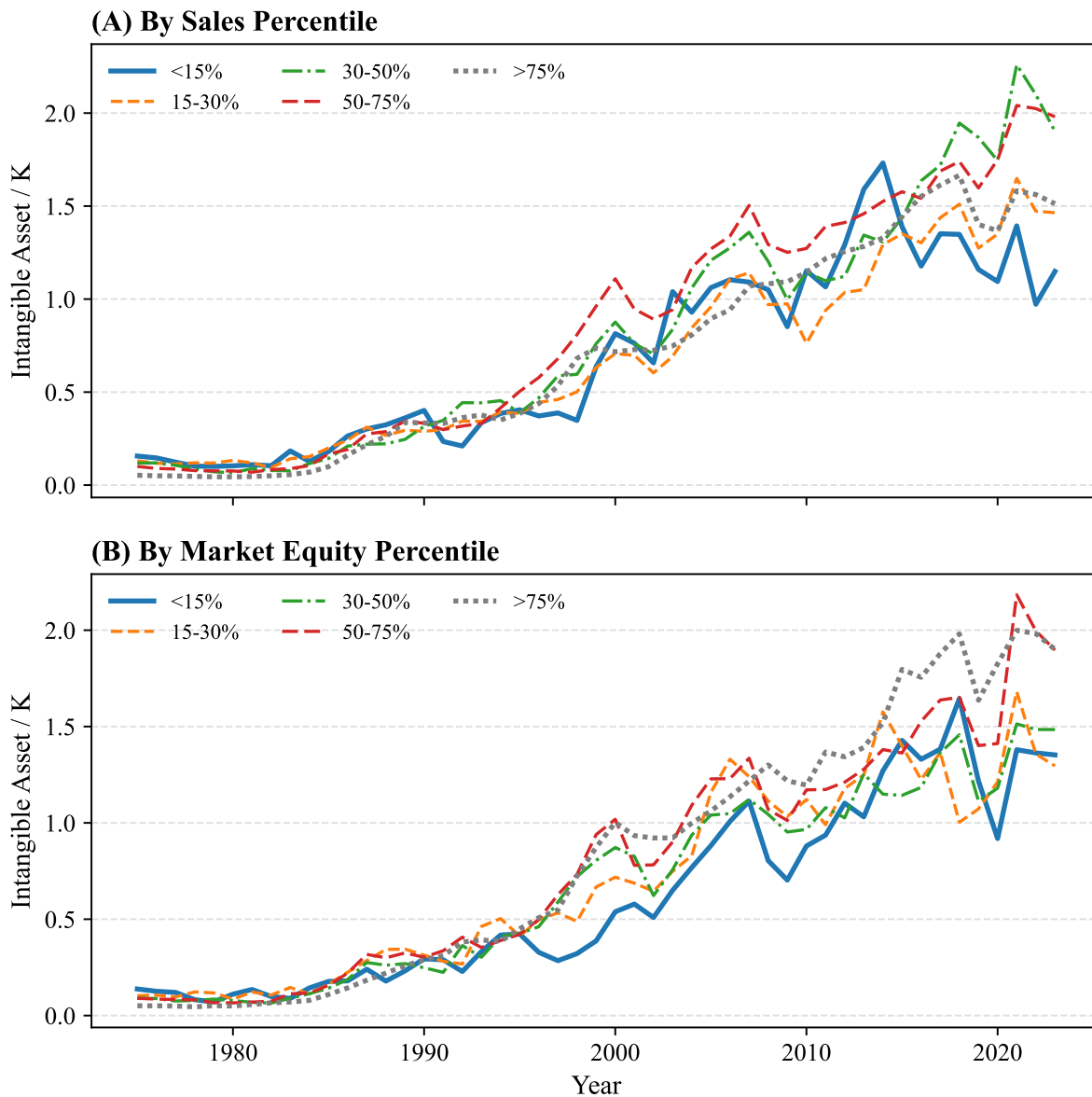


Figure 6: Intangible Assets by Firm Size. This figure presents the aggregate intangible asset (Intang) to physical (K) capital ratio by size over time. Size is measured by sales or market equity, and the ratio is measured as the total intangible assets (INTAN) divided by PPEGT. For each year, firms are categorized into five groups based on their sizes, and we estimate the intangible assets to physical capital ratio within each group for a given year. Source: Compustat Fundamentals Annual for 1975-2023.

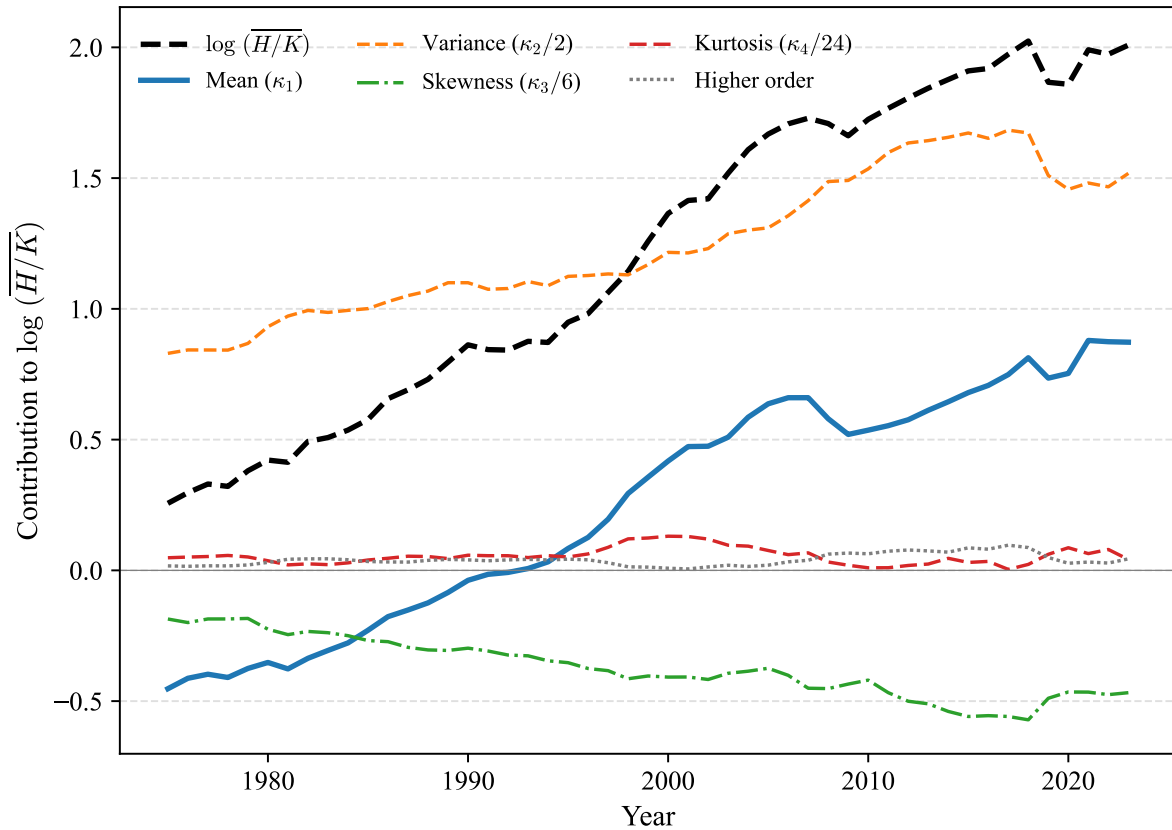


Figure 7: Cumulant Decomposition of Aggregate Intangible Intensity. This figure decomposes aggregate intangible intensity H_{it}/K_{it} into contributions from the cross-sectional cumulants of $x_{it} \equiv \log(H_{it}/K_{it})$: the mean (κ_1), variance ($\kappa_2/2!$), skewness ($\kappa_3/3!$), kurtosis ($\kappa_4/4!$), and higher-order terms. The dashed black line plots the actual $\log(1/N \sum_i H_{it}/K_{it})$. Source: Compustat Fundamentals Annual for 1975–2023.

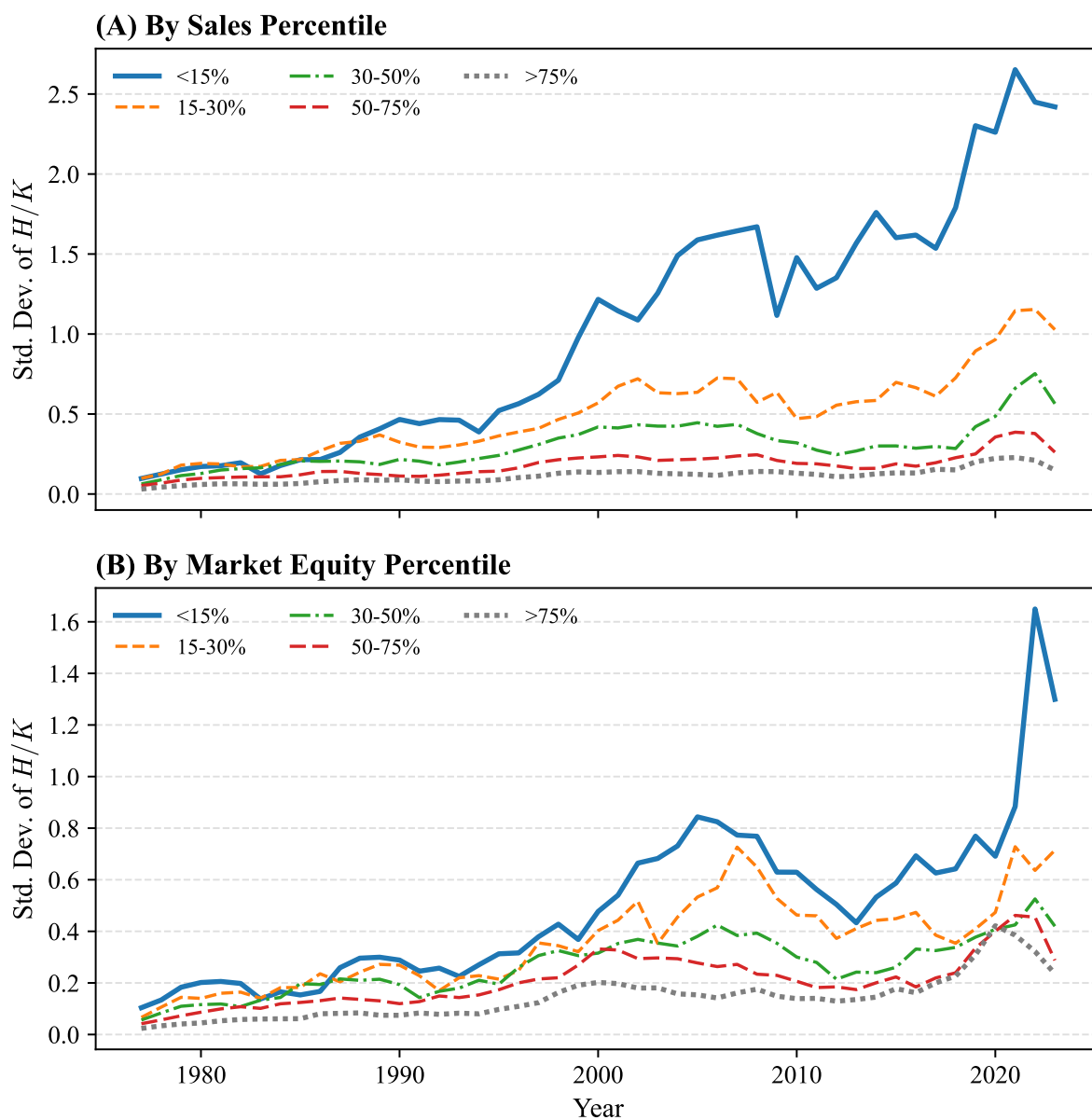


Figure 8: Intangible Volatility by Firm Size. This figure presents the firm-level volatility of intangible (H) to physical (K) capital ratio ($hkratio$) by size over time. Size is measured by sales and market equity. The volatility of $hkratio$ is measured at the firm level. For each firm j year t , we estimate the standard deviation of the $hkratio_{i,t}$ over the last 5 years. The estimation is done over a rolling window. For each year, firms are categorized into five groups based on their sizes, and we estimate the mean $hkratio$ volatility within each group for a given year. Source: Compustat Fundamentals Annual for 1975-2023.

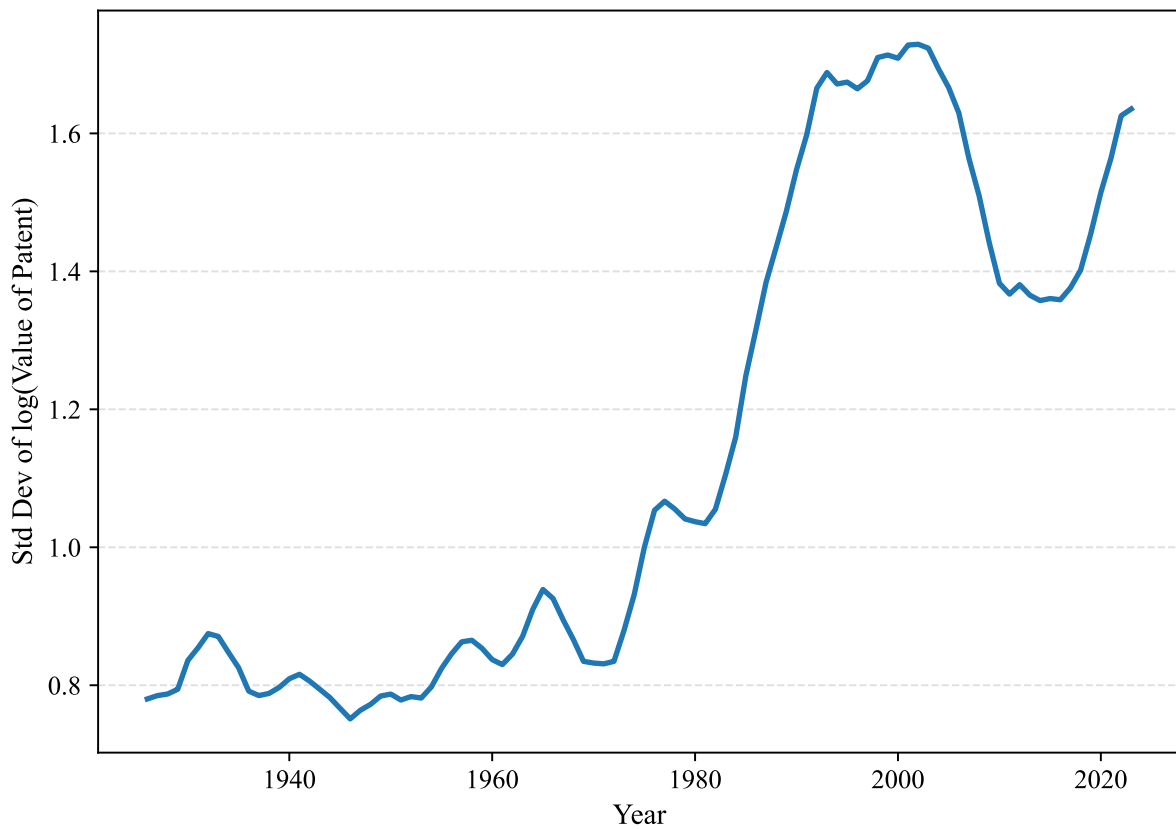


Figure 9: Standard Deviation of Log(Value of Patent) by Year. This figure plots the time series of the cross-sectional dispersion in patent value using the patent-level data. The vertical axis shows the standard deviation of the logarithm of normalized patent value, computed within a rolling three-year window. For each year t , the dispersion is calculated using all patents granted in years $t - 2$ through t . The patent value measure is obtained directly from [Kogan et al. \(2017\)](#), who construct market-based valuations of U.S. patents using stock market reactions to patent grant announcements. Each patent's value is normalized by the median within the same CPC subclass and year to account for technological field heterogeneity. The 1975 standard deviation is normalized to one.

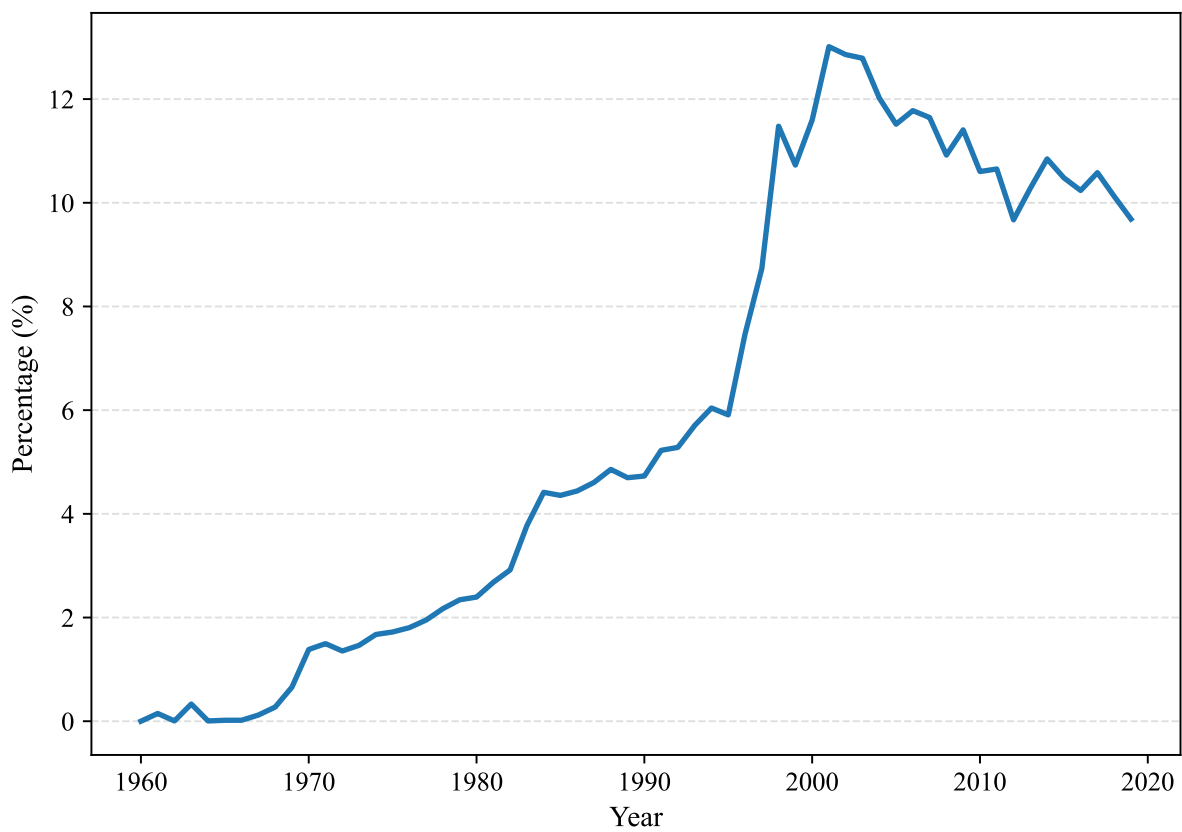


Figure 10: Share of Equity Allocated to Employee Incentives by Year. This figure shows the annual percentage of total corporate equity allocated to employee ownership and stock-based compensation. The firm-level employee stock-based compensation data are obtained from [Eisfeldt, Falato, and Xiaolan \(2023\)](#).

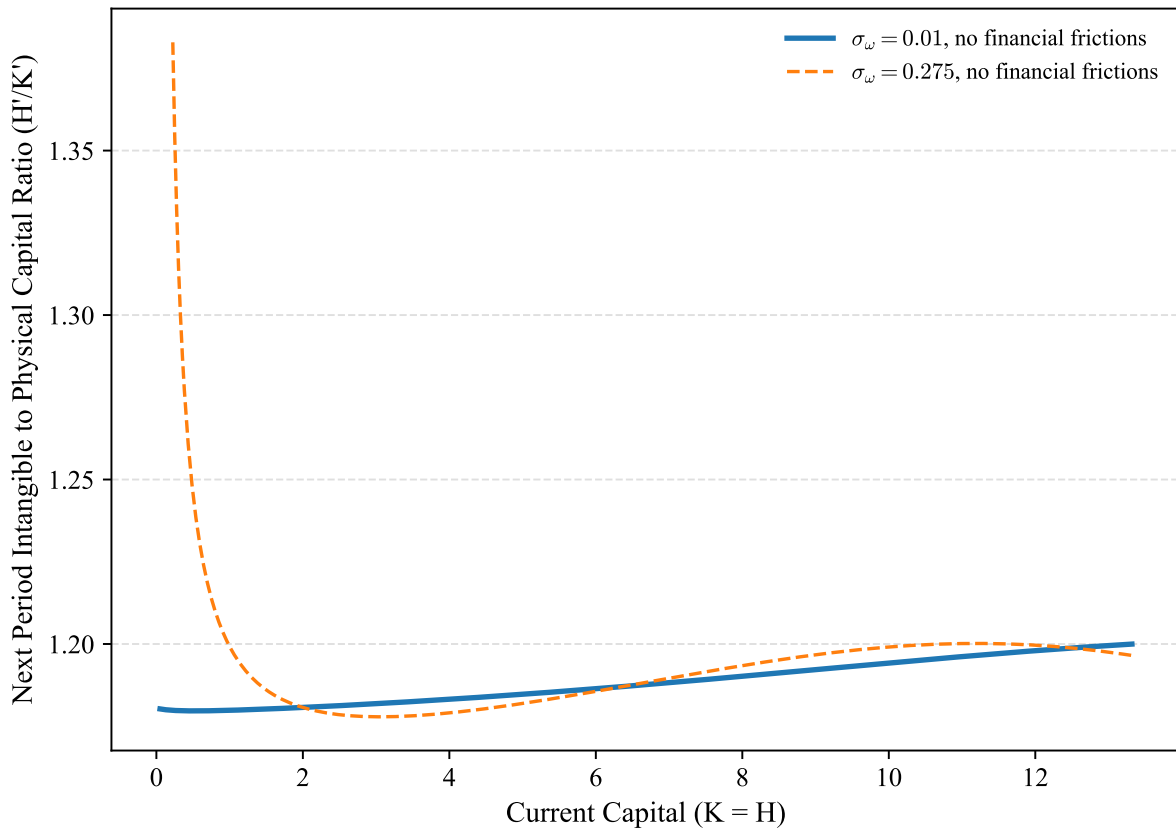
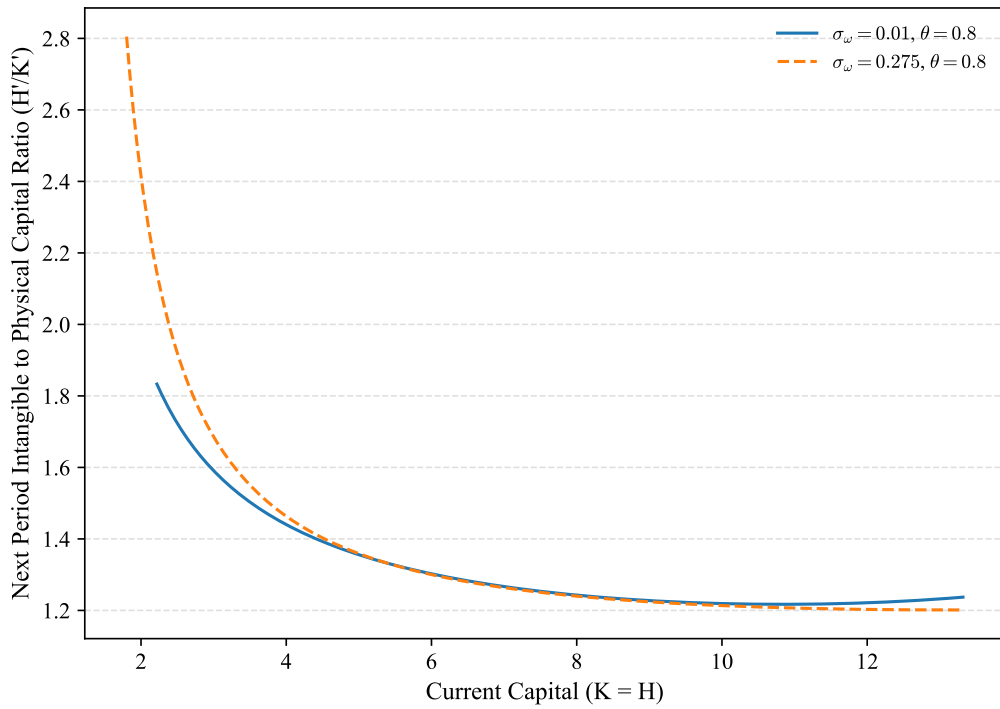
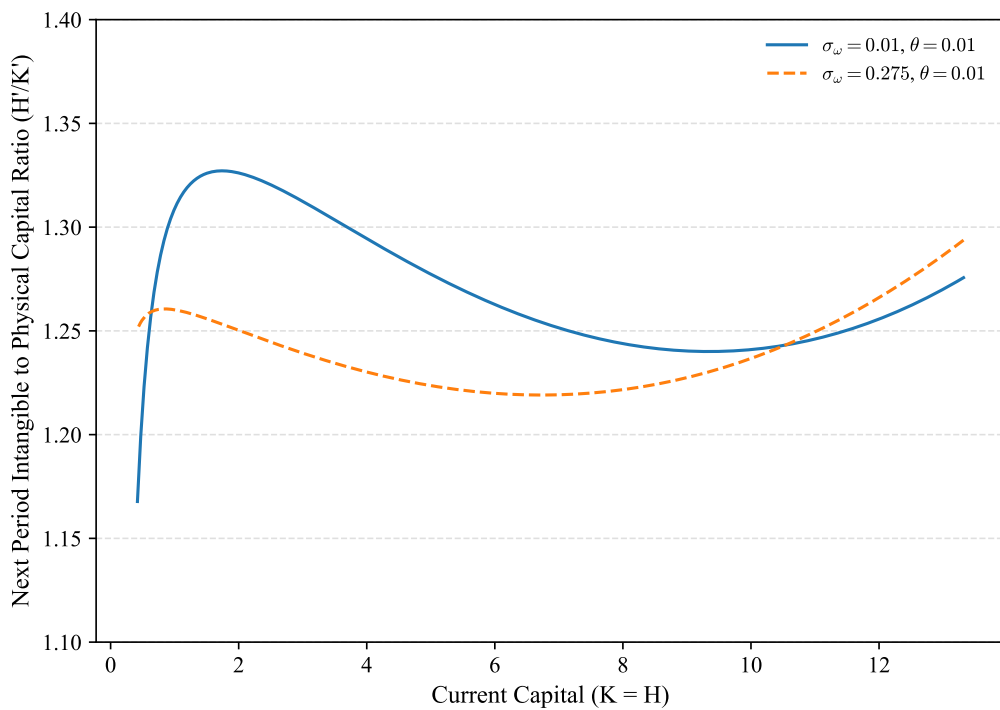


Figure 11: Intangible Investment and Uncertainty without Financial Frictions. This figure plots the ratio of next-period intangible to physical capital H'/K' as a function of current capital ($K = H$) under two levels of quality-shock volatility, $\sigma_\omega = 0.01$ and $\sigma_\omega = 0.275$, in an environment without financial frictions.



(a) Benchmark financing capacity ($\theta = 0.8$)



(b) Severe financing frictions ($\theta = 0.01$)

Figure 12: Financing Frictions and Quality Shock. Both panels plot the optimal next-period intangible-to-physical capital ratio H'/K' along the diagonal $K = H$ for two levels of quality-shock volatility, $\sigma_\omega = 0.01$ and $\sigma_\omega = 0.275$. Panel (a): benchmark financing capacity ($\theta = 0.8$). Panel (b): severe financing frictions ($\theta = 0.01$).⁵⁰

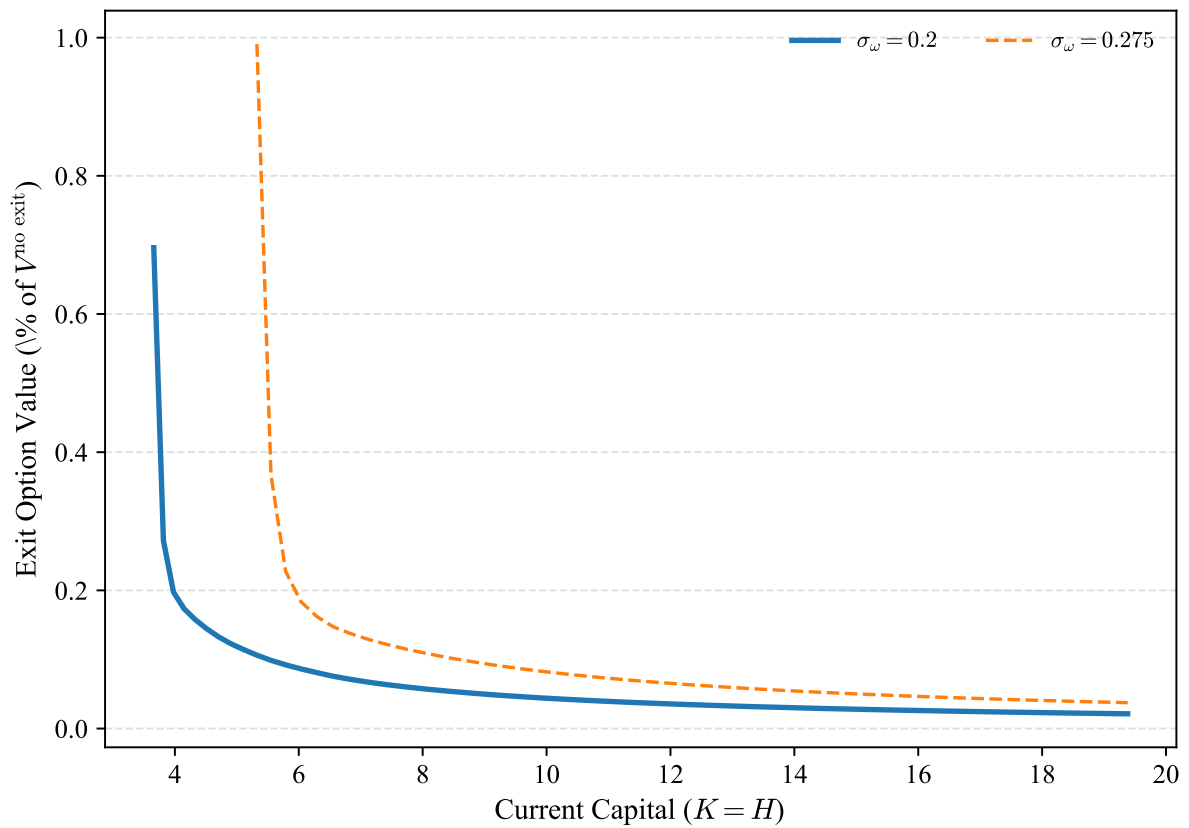


Figure 13: Value of Exit Option. This figure plots the percentage change in the firm's value attributable to the exit option as a function of current capital, with physical and intangible capital set equal ($K = H$). The y-axis reports $(V^{\text{exit}} - V^{\text{no exit}}) / V^{\text{no exit}}$, where V^{exit} denotes the value function with the exit option and $V^{\text{no exit}}$ denotes the value function without the exit option.

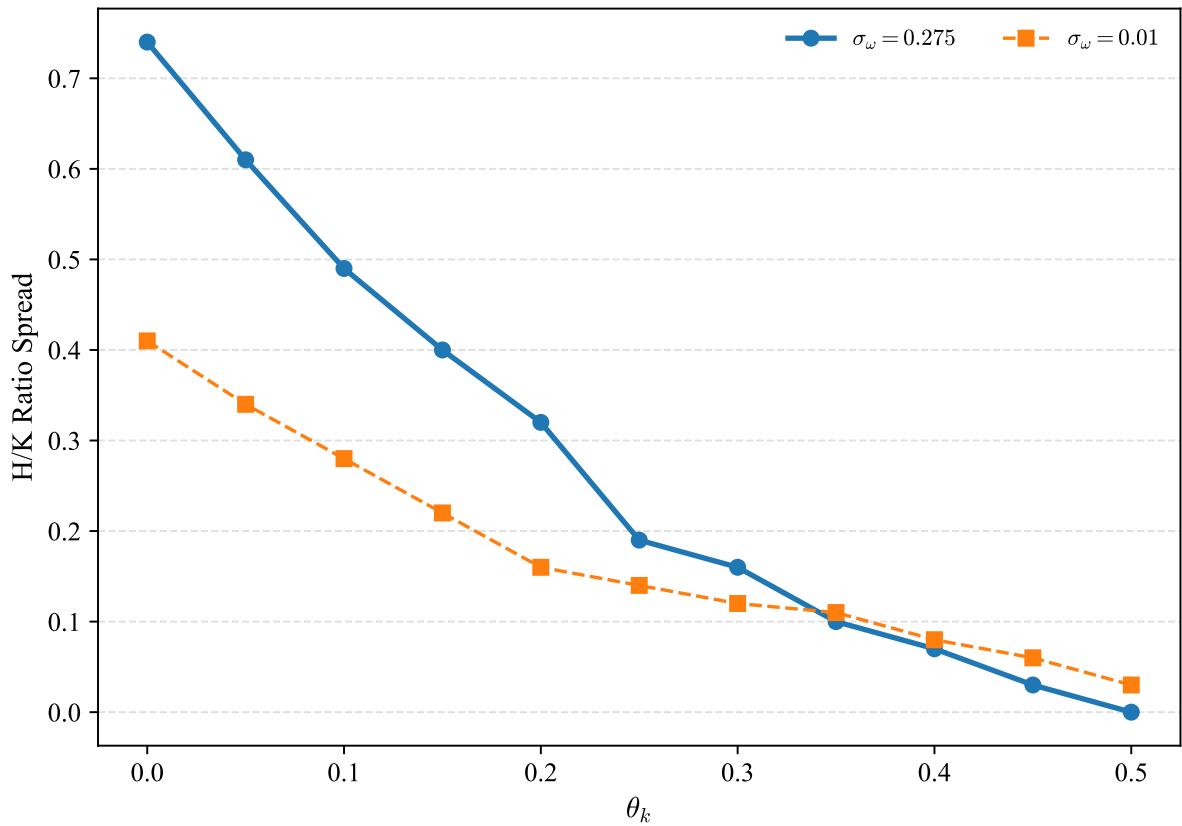


Figure 14: Intangible Intensity Dispersion and the Collateral Role of Physical Capital. This figure plots the H/K spread, defined as the difference in the intangible-to-physical capital ratio between small firms (below the 15th percentile) and large firms (above the 75th percentile of the size distribution), as a function of the pledgeability of physical capital, θ_k . The pledgeability of intangible capital is fixed at $\theta_h = 0.8$. The blue line corresponds to an economy with quality volatility ($\sigma_\omega = 0.275$), while the yellow line shows the counterpart economy without quality shocks.

Table 1: Summary Statistics. This table reports summary statistics for key investment, and financial condition variables across firm size groups. Firm size is defined based on the distribution of market equity (ME). All variables are expressed in real terms and deflated using the Consumer Price Index (CPI). The sample includes U.S. public firms from Compustat Annual over the period 1975–2023.

	Size				
Market Equity	<15%	15–30%	30–50%	50–75%	>75%
Panel A: Firm Dynamics					
AT (real)	18.52	56.55	147.36	437.84	17011.93
Age (Found)	20.53	23.67	31.86	44.03	84.63
Labor Growth (%)	3.17	3.48	3.37	2.63	1.06
Asset Growth (%)	2.34	3.62	3.83	3.30	1.92
Std of Asset Growth (%)	7.98	6.59	6.05	5.47	4.81
Std of Labor Growth (%)	11.88	9.80	9.06	7.97	6.32
Tobin's q	2.82	2.48	2.25	2.11	1.94
ROA (%)	-8.27	6.19	10.28	11.96	13.29
Panel B: Investment					
H(SGA)/K	1.93	1.48	1.21	0.93	0.77
H(RD)/K	1.35	0.95	0.76	0.49	0.37
Stkco / ME (%)	1.71	1.24	1.00	0.78	0.48
Stkco / AT (%)	5.69	3.70	2.22	1.60	0.85
R&D / AT (%)	13.96	7.54	4.82	3.30	3.43
SGA / AT (%)	47.61	37.38	31.92	29.14	23.98
AD / AT (%)	1.70	1.77	1.84	1.71	2.10
Patent Value / AT (%)	66.93	24.21	15.22	13.31	7.18
Panel C: Financial Conditions					
DIV	0.19	0.32	0.45	0.59	0.88
SA	-2.44	-2.95	-3.31	-3.70	-4.30
WW	-0.14	-0.20	-0.25	-0.31	-0.46
Book Leverage (%)	41.81	39.11	42.92	48.62	56.80
Market Leverage (%)	17.74	20.54	23.05	27.13	20.54

Table 2: Intangible-Physical Capital Ratio and Firm Size. This table summarizes the results from running the following regression:

$$\frac{H_{i,t}}{K_{i,t}} = \beta_0 + \beta_1 \log(\text{size}_{i,t}) + \alpha_i + \delta_t.$$

This table reports firm-level regressions of different measures of the intangible-to-physical capital ratio (H/K) on firm size. The dependent variable represents different measures of intangible capital intensity, including total intangible capital, SG&A-based organizational capital, and R&D-based innovation capital. Firm size is measured alternatively by the logarithm of sales (Columns 1–3) and the logarithm of market equity (Columns 4–6). The sample period is 1975–2023. Standard errors are clustered at firm level and are collected in parentheses. * $p < 0.1$; ** $p < 0.05$; *** $p < 0.01$.

	By Sales			By Market Equity		
	Total (1)	R&D (2)	SG&A (3)	Total (4)	R&D (5)	SG&A (6)
log(sales)	-2.248*** (0.303)	-1.181*** (0.174)	-1.199*** (0.231)			
log(me)				-1.237*** (0.161)	-0.770*** (0.124)	-0.691*** (0.089)
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
Adj R^2	0.274	0.252	0.215	0.276	0.260	0.212
Observations	257,238	257,238	257,238	225,426	225,426	225,426

Table 3: Parameter Values: Benchmark. This table reports the parameter values of our benchmark calibration.

Parameter	Symbol	Value
Discount factor	β	0.96
Physical capital share	α_k	0.3
Intangible capital share	α_h	0.4
Depreciation of physical capital	δ_k	0.1
Depreciation of intangible capital	δ_h	0.1
Intangible capital financing capacity	θ	0.8
Fixed production cost	ξ	2.45
Adjustment cost of physical capital	ρ_k	0.1
Adjustment cost of intangible capital	ρ_h	0.1
Standard deviation of intangible quality shock	σ_ω	0.2
Idiosyncratic productivity shock persistence	ρ_s	0.8
Idiosyncratic productivity shock volatility	σ_s	0.1

Table 4: Model Fit: Data and Size Moments. The model is calibrated using U.S. firm-level data over the period 1975–2000 to match moments that capture firms’ physical and intangible capital dynamics. Panel A reports moments from the 1975-2000 sample and from the model simulated under parameter values ($\theta = 0.8, \sigma_\omega = 0.2$), including the firm exit rate, the H/K spread, the share of employee equity in firm value, and the average firm age. H/K spread = $(H/K)_{size < 15\%} - (H/K)_{size > 75\%}$, where firm size is measured by market equity. Panel B evaluates the model’s performance in replicating the cross-sectional pattern of intangible intensity across firm size groups, comparing the H/K ratios across size quantiles in the data and in the model.

Panel A: Data and Model Statistics					
	Data: 1975-2000	$(\theta, \sigma_\omega) =$ (0.8, 0.2)	$(\theta, \sigma_\omega) =$ (0.8, 0.01)	$(\theta, \sigma_\omega) =$ (0.01, 0.2)	$(\theta, \sigma_\omega) =$ (0.01, 0.01)
Exit rate	0.046	0.035	0.022	0.215	0.261
H/K spread	0.47	0.50	0.20	-0.25	-0.29
Investment rate of K	0.10	0.14	0.13	0.01	-0.04
Investment rate of H	0.11	0.14	0.12	0.02	-0.05
Employee equity to value	0.05	0.11	0.07	0.01	0.01
Age	44	58	78	173	129

Panel B: Intangible Investment by Size					
	<15%	[15%,30%]	[30%,50%]	[50%,75%]	>75%
H/K Ratio Data	1.676	1.437	1.374	1.336	1.209
H/K Ratio $\theta = 0.8, \sigma_\omega = 0.2$	1.818	1.375	1.348	1.333	1.315
H/K Ratio $\theta = 0.8, \sigma_\omega = 0.01$	1.541	1.340	1.340	1.340	1.340
H/K Ratio $\theta = 0.01, \sigma_\omega = 0.2$	1.123	1.158	1.355	1.327	1.370
H/K Ratio $\theta = 0.01, \sigma_\omega = 0.01$	1.060	1.285	1.201	1.354	1.347

Table 5: Comparative Statics: Effects of Quality Shock Volatility and Financing Capacity.

This table examines how quality shock volatility (σ_ω) and financing capacity (θ) affect firms' intangible intensity. Each row represents a different calibration scenario, varying σ_ω and θ to isolate the effects of the real-option and financing channels. The first column reports the model-generated H/K spread, defined as $(H/K)_{<15\%} - (H/K)_{>75\%}$ based on firm size measured by market equity. The second column shows the ratio of employee equity to firm value, and the third column presents the ratio of the model-implied H/K spread to the empirical counterpart. Higher σ_ω increases the dispersion in intangible intensity through the real-option mechanism.

	H/K spread	Employee equity to value	% of data (H/K spread)
Data: 1975-2000	0.467	4.8%	100%
Low $\sigma_\omega = 0.2$, low $\theta = 0.8$	0.503	10.5%	107.7%
Low $\sigma_\omega = 0.2$, high $\theta = 1.2$	0.160	16.2%	34.3%
Data: 2001-2023	1.374	11.1%	100%
High $\sigma_\omega = 0.275$, low $\theta = 0.8$	0.812	18.7%	60.1%

A Data Appendix

A.1 Sample Construction

Our primary firm-level data come from Compustat Fundamentals Annual, covering fiscal years 1975 through 2023. We obtain replacement-cost measures of firms' intangible capital from [Peters and Taylor \(2017\)](#) (henceforth PT), distributed via Wharton Research Data Services (WRDS), which provides knowledge capital, organization capital, off-balance-sheet intangibles, and total Tobin's q . We measure firm-level patent market values using the 2023 vintage of the [Kogan et al. \(2017\)](#) (KPSS) dataset. We obtain the Consumer Price Index (CPI) from the U.S. Bureau of Labor Statistics. The aggregate time series of equity allocated to employee stock-based compensation used in Section 2.3.3 comes from [Eisfeldt, Falato, and Xiaolan \(2023\)](#).

We restrict the sample to U.S.-incorporated firms and exclude regulated utilities (SIC 4900–4999), financial firms (SIC 6000–6999), and firms in public service or non-operating categories (SIC ≥ 9000). We drop firm-year observations with non-positive total assets (`at`), non-positive sales (`sale`), negative capital expenditures (`capx`), negative R&D expenses (`xrd`), or non-positive net or gross property, plant, and equipment (`ppent`, `ppegt`). We further drop observations with missing knowledge capital, organization capital, or `ppegt`, as these are essential inputs for the intangible-to-physical capital ratio. All nominal variables are deflated using the CPI with 1990 as the base year. To limit the impact of outliers, we winsorize all level variables at the 1% and 99% percentiles.

A.2 Measuring Intangible Capital

We follow PT in measuring the replacement cost of a firm's intangible capital, K_{it}^{int} , as the sum of three components: knowledge capital G_{it} , organization capital S_{it} , and externally purchased intangibles reported on the balance sheet (Compustat `intan`, which includes Goodwill). The first two components are constructed by applying the perpetual-inventory method to firms'

intangible investments,

$$H_{it} = (1 - \delta_H)H_{i,t-1} + I_{H,it}, \quad H \in \{G, S\}, \quad (14)$$

where $I_{G,it} = \text{xrd}_{it}$ is investment in knowledge capital, and $I_{S,it} = 0.3 \cdot \text{SG\&A}_{it}^{\text{net}}$ is investment in organization capital. The 30% multiplier follows [Corrado, Sichel, and Hulten \(2004\)](#), [Eisfeldt and Papanikolaou \(2013\)](#), and PT, who interpret the remaining 70% of SG&A as operating expenses supporting current-period production. The depreciation rates are set to BEA industry-specific values for δ_G , ranging from 10% for pharmaceuticals to 40% for computers and peripheral equipment, and $\delta_S = 20\%$ following [Falato et al. \(2022\)](#).

A subtle measurement issue arises when constructing net SG&A. As detailed in PT Appendix B.1, the Compustat variable `xsga` typically already includes `xrd`, so net SG&A must be computed as `xsga - xrd - rdip`, where `rdip` captures in-process R&D from acquisitions. The PT dataset we obtain from WRDS incorporates this adjustment, as well as PT's procedure for estimating each firm's initial capital stock $H_{i,0}$ from pre-IPO data and founding-year information.

The total intangible capital stock is $K_{it}^{\text{int}} = G_{it} + S_{it} + \text{intan}_{it}$, and physical capital is $K_{it}^{\text{phy}} = \text{ppeg}_{it}$. Our key ratio of interest is

$$hkratio_{it} = \frac{K_{it}^{\text{int}}}{K_{it}^{\text{phy}}}.$$

We also construct analogous ratios using only knowledge capital ($G_{it}/K_{it}^{\text{phy}}$) or only organization capital ($S_{it}/K_{it}^{\text{phy}}$) for the robustness analysis in Figure 3.

A.3 Patent Value

We construct firm-year measures of patent market value using the 2023 vintage of the KPSS dataset. For each granted patent, KPSS provide a dollar-value estimate ξ_p^{real} derived from the abnormal stock-market return around the patent's grant date. We aggregate to the firm-year level.

For Figure 9, we construct a measure of cross-sectional patent-value dispersion that controls for technological-field heterogeneity. We first normalize each patent's value by the median value within the same (CPC subclass, issue year) cell, take the natural logarithm of the normalized value, and compute its standard deviation within a rolling three-year window. The 1975 value is normalized to one to facilitate comparison across years.

B Proofs

B.1 Proof of Proposition 1

The firm's objective in (3) admits the closed form

$$U(h_{t+1}, \sigma) \equiv h_{t+1}^\alpha N(d_1) - K(h_{t+1})N(d_2),$$

where $K(h_{t+1}) = h_{t+1} - h_t + \xi$, and d_1, d_2 are as defined in the main text. We use ϕ and N to denote the standard normal pdf and cdf, and we drop the time subscripts on h when there is no ambiguity. The Black–Scholes–Merton identity $h^\alpha \phi(d_1) = K(h)\phi(d_2)$, together with $\partial d_1/\partial h = \partial d_2/\partial h$, implies that the first-order condition simplifies to

$$U_h(h_{t+1}^*, \sigma) = \alpha (h_{t+1}^*)^{\alpha-1} N(d_1) - N(d_2) = 0. \quad (15)$$

The second-order condition $U_{hh}(h_{t+1}^*, \sigma) < 0$ holds at the maximum. By the implicit function theorem,

$$\frac{\partial h_{t+1}^*}{\partial \sigma} = - \frac{U_{h\sigma}(h_{t+1}^*, \sigma)}{U_{hh}(h_{t+1}^*, \sigma)},$$

so the sign of $\partial h_{t+1}^*/\partial \sigma$ equals the sign of $U_{h\sigma}$.

Differentiating (15) with respect to σ ,

$$U_{h\sigma} = \alpha h^{\alpha-1} \phi(d_1) \frac{\partial d_1}{\partial \sigma} - \phi(d_2) \frac{\partial d_2}{\partial \sigma}.$$

Direct computation yields the identities

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_2}{\sigma}, \quad \frac{\partial d_2}{\partial \sigma} = -\frac{d_1}{\sigma}.$$

Applying $h^\alpha \phi(d_1) = K(h)\phi(d_2)$ then gives

$$U_{h\sigma} = \frac{\phi(d_2)}{\sigma} \left[d_1 - \frac{\alpha K(h)}{h} d_2 \right].$$

At h_{t+1}^* , the FOC (15) implies $\alpha(h_{t+1}^*)^{\alpha-1} = N(d_2)/N(d_1)$, equivalently

$$\frac{\alpha K(h_{t+1}^*)}{h_{t+1}^*} = \frac{\phi(d_1)}{\phi(d_2)} \cdot \frac{N(d_2)}{N(d_1)},$$

where we used the BSM identity once more to rewrite $K/h^\alpha = \phi(d_1)/\phi(d_2)$. Substituting,

$$\text{sgn}(U_{h\sigma}|_{h_{t+1}^*}) = \text{sgn}(d_1 N(d_1)\phi(d_2) - d_2 N(d_2)\phi(d_1)).$$

Define $f(x) \equiv xN(x)/\phi(x)$. The right-hand side has the same sign as $f(d_1) - f(d_2)$. Differentiating,

$$f'(x) = \frac{N(x)(1+x^2) + x\phi(x)}{\phi(x)}.$$

The Mills-ratio inequality implies $N(x)(1+x^2) + x\phi(x) > 0$ for all $x \in \mathbb{R}$, so $f'(x) > 0$ globally. Since $d_1 = d_2 + \alpha\sigma > d_2$, we have $f(d_1) > f(d_2)$, hence $U_{h\sigma}|_{h_{t+1}^*} > 0$ and $\partial h_{t+1}^*/\partial \sigma > 0$. \square

B.2 Proof of Proposition 2

The exit option value is $V_t - V_{0,t}$, where V_t is the value with the option to exit and $V_{0,t}$ is the value with the firm forced to operate in period $t+1$. We compute their derivatives with respect to h_t separately.

For $V_{0,t}$, the FOC of the unconstrained problem $V_{0,t} = \max_{h_{t+1}} h_{t+1}^\alpha - K(h_{t+1})$ is $\alpha(h_{t+1}^*)^{\alpha-1} =$

1, which yields $h_{t+1}^{**} = \alpha^{1/(1-\alpha)}$. Since h_{t+1}^{**} does not depend on h_t , substituting back gives

$$V_{0,t} = (h_{t+1}^{**})^\alpha - (h_{t+1}^{**} - h_t) - \xi,$$

so $\partial V_{0,t}/\partial h_t = 1$.

For V_t , the variable h_t enters the firm's optimization only through the cost function $K(h_{t+1}) = h_{t+1} - h_t + \xi$, with $\partial K/\partial h_t = -1$. By the envelope theorem,

$$\frac{\partial V_t}{\partial h_t} = E_t \left[-\frac{\partial K}{\partial h_t} \mathbf{1}\{\omega_{t+1} \geq \omega^*\} \right] = \Pr(\omega_{t+1} \geq \omega^*) = N(d_2),$$

where ω^* is the exit threshold and d_2 is evaluated at h_{t+1}^* .

Combining the two derivatives,

$$\frac{\partial (V_t - V_{0,t})}{\partial h_t} = N(d_2) - 1 < 0,$$

since $N(d_2) \in (0, 1)$. The exit option value is therefore strictly decreasing in h_t . \square

B.3 Proof of Proposition 3

In this appendix, we provide the proof for the existence, uniqueness, and convergence of the invariant distribution of firms, μ^* . The proof relies on the theoretical framework established by Stokey, Lucas, and Prescott (1989) (hereafter SLP89).

Let $S \subset \mathbb{R}_{++}$ be the set of idiosyncratic productivity shocks, $K \subset \mathbb{R}_+$ be the set of physical capital, and $H \subset \mathbb{R}_+$ be the set of pre-shock intangible capital. The state space for an individual firm is defined as $X = S \times K \times H$. We assume X is compact, bounded by sufficiently large upper bounds on capital and productivity. Let \mathcal{X} be the Borel σ -algebra on X , and $\mathcal{M}(X)$ be the space of probability measures on (X, \mathcal{X}) .

The industry dynamics are driven by a transition function $P : X \times \mathcal{X} \rightarrow [0, 1]$. For a firm with current state $x = (s, k, \tilde{h})$, the transition function is determined by the optimal policy functions for capital accumulation, $k'(x)$ and $\tilde{h}'(x)$, the exit rule $\chi(x, \omega)$, and the stochastic processes for productivity s' and intangible quality ω' .

The law of motion for the distribution of firms, $\mu_{t+1} = \Psi(\mu_t)$, is defined by the mapping $\Psi : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$:

$$\mu_{t+1}(\mathcal{B}) = \int_X P(x, \mathcal{B}) \mu_t(dx) + \nu(\mathcal{B}), \quad \forall \mathcal{B} \in \mathcal{X} \quad (16)$$

where ν represents the measure of new entrants.

The transition function P has the Feller property. That is, for any bounded continuous function $f : X \rightarrow \mathbb{R}$, the function defined by $Tf(x) = \int_X f(x') P(x, dx')$ is continuous in x .

The firm chooses to continue operating if the realized intangible quality shock ω satisfies $\omega \geq \omega^*(s, k, \tilde{h})$, where the threshold ω^* is defined by the zero-value condition $V(s, k, \omega^* \tilde{h}) = 0$. Since the value function V is continuous and strictly increasing in the effective intangible capital $\omega \tilde{h}$, the implicit function theorem implies that the threshold function $\omega^*(x)$ is continuous in x .

The expectation operator $Tf(x)$ can be expressed as an integration over the future productivity shock s' and the quality shock ω' :

$$Tf(x) = \int_S \int_{\omega^*(x)}^{\infty} f(s', k'(x), \tilde{h}'(x)) dH^\omega(\omega') dH^s(s'|s) \quad (17)$$

where H^ω and H^s denote the cumulative distribution functions for the log-normal quality shock and the AR(1) productivity shock, respectively.

Under standard assumptions of convex adjustment costs and strict concavity of the production function, the Theorem of the Maximum ensures that the policy functions $k'(x)$ and $\tilde{h}'(x)$ are continuous. Furthermore, the probability density functions of s' and ω' are continuous. Although the domain of integration is truncated by $\omega^*(x)$, the continuity of the threshold $\omega^*(x)$ and the continuity of the underlying distributions ensure that the integral varies continuously with the state x . Thus, P satisfies the Feller property.

Existence

Since the state space X is compact and the transition function P satisfies the Feller property, Theorem 12.10 in SLP89 guarantees the existence of an invariant probability measure μ^* such that $\mu^* = \Psi(\mu^*)$.

Uniqueness and Convergence

To establish uniqueness and global convergence, we invoke Theorem 12.12 in SLP89, which requires the transition function to satisfy *monotonicity* and a *mixing condition*.

Monotonicity. Monotonicity requires that if $x_a \geq x_b$ (component-wise), then the distribution $P(x_a, \cdot)$ stochastically dominates $P(x_b, \cdot)$. In our model, larger firms (higher s, k, \tilde{h}) possess higher continuation values, leading to a lower exit threshold ω^* and a higher probability of survival. Additionally, due to the complementarity of inputs and convex adjustment costs, the optimal investment policies $k'(x)$ and $\tilde{h}'(x)$ are non-decreasing in the current state variables. Consequently, a firm starting with a higher state vector is likely to transition to a higher state vector in the subsequent period, satisfying the monotonicity condition.

Mixing Condition. Following [Cooley and Quadrini \(2001\)](#), we verify the mixing condition by showing that there exists a state $\hat{x} \in X$, a probability $\varepsilon > 0$, and an integer $n \geq 1$ such that the system can transition from the lower bound of the state space to the upper region, and from the upper bound to the lower region after n periods.

Let \underline{x} and \bar{x} denote the infimum and supremum of the compact state space X . We partition X into a "lower set" $X_L = [\underline{x}, \hat{x}]$ and an "upper set" $X_U = [\hat{x}, \bar{x}]$. The mixing condition requires:

$$\Psi^n(\underline{x}, X_U) \geq \varepsilon \quad \text{and} \quad \Psi^n(\bar{x}, X_L) \geq \varepsilon \quad (18)$$

In our model, the intangible quality shock ω follows a log-normal distribution, and the productivity shock s follows a Gaussian AR(1) process. Since the probability density functions of these shocks are strictly positive over their entire support $(0, \infty)$ and $(-\infty, \infty)$ respectively, there

is a strictly positive probability that a firm starting at the lowest state \underline{x} receives a sequence of sufficiently large positive shocks to accumulate capital and reach the set X_U in N periods. Conversely, a firm at the highest state \bar{x} faces a strictly positive probability of receiving a sequence of negative shocks (or low ω) that leads to decumulation or exit, reaching the set X_L .

Since $h^\omega(\omega) > 0$ and $h^s(s'|s) > 0$ everywhere, this condition is satisfied for any interior splitting point \hat{x} with sufficient n , analogous to the argument in [Cooley and Quadrini \(2001\)](#).

Since P is monotone and satisfies the mixing condition, Theorem 12.12 in SLP89 applies. Therefore, the invariant distribution μ^* is unique, and for any initial distribution μ_0 , the sequence $\{\Psi^n(\mu_0)\}$ converges weakly to μ^* . □